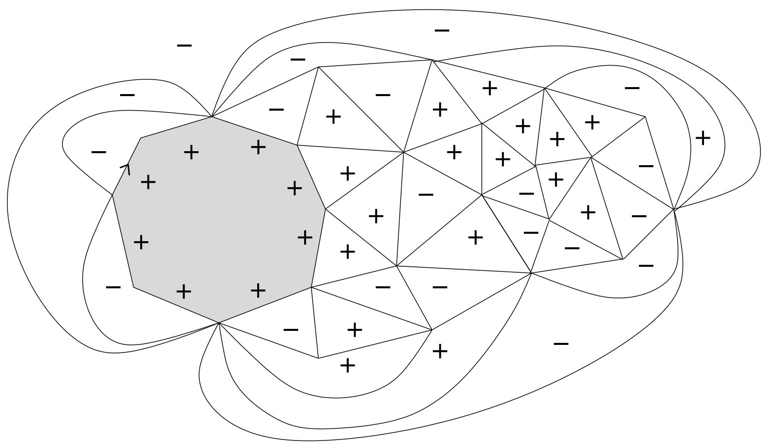
Chapter VIII

Ising model

In statistical physics, the Ising model represents a simplified model for magnetization. Each piece of the surface (here each face of the map) carries a unit of magnetization, pointing either upward $+$ or downward $-$. This can also be represented as a map with bicolored faces black/white, or $+/-$, or any other convenient choice. The color is also called the spin, worth $+$ or $-$.

Our goal is to put the Ising model on a random map, i.e. study the generating functions counting bi-colored maps.



In this chapter, we extend the previous method of Tutte's equations and its solution by topological recursion, to bicolored maps, i.e. Ising model maps. The method is more or less the same: define generating functions as formal series in t^v where v is the number of vertices, then write loop equations (generalization of Tutte's equations), and then solve loop equations.

The loop equation for the disc, is an algebraic equation, and thus the disc amplitude is an algebraic function, called the "spectral curve".

Then, once we know the spectral curve, all the other amplitudes can be computed by the topological recursion of chapter VII. It may look surprising that the same topological recursion which solves the loop equations for non–colored maps, also solves the more intricated loop equations of the Ising model. In fact, this same topological recursion also solves the loop equations of many other enumerative geometry problems, like the $O(\mathfrak{n})-$model on a random map.

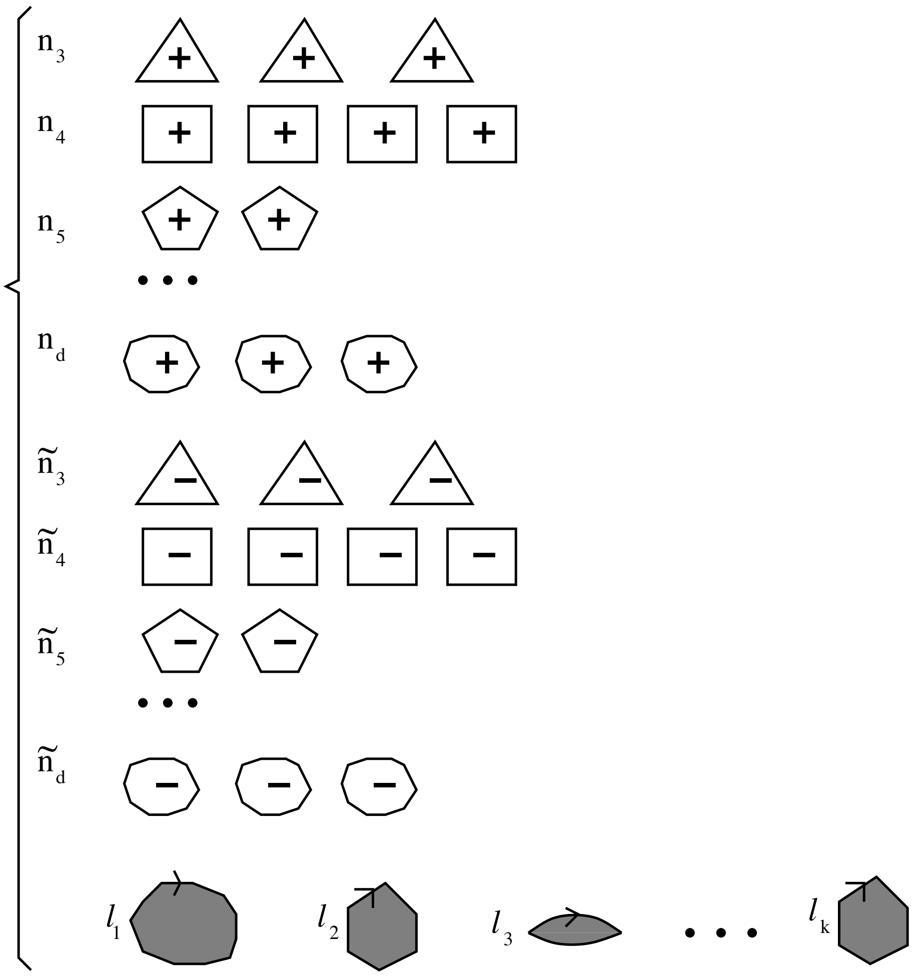
In this chapter, we don't present all computations in details, we just give the definitions of the model, and the Tutte-like equations, and then the solution without detailed proof.

The main new feature compared to maps, is that we also compute generating functions for maps having fixed spins boundary conditions on their marked faces, that is multi–colored boundaries. For such boundary generating functions, we merely state the main results, without proofs (proofs are in the literature).

1 Bicolored maps

The Ising model is a model of maps carrying 2 possible "colors" or 2 possible "spins" $+$ or $-$. Or, let us say, the unmarked faces can carry a spin $+$ or $-$. Here, spin means just color, the spin can take two values $+$ or $-$.

Our maps are constructed by gluing the following sorts of oriented polygonal pieces, marked on unmarked:

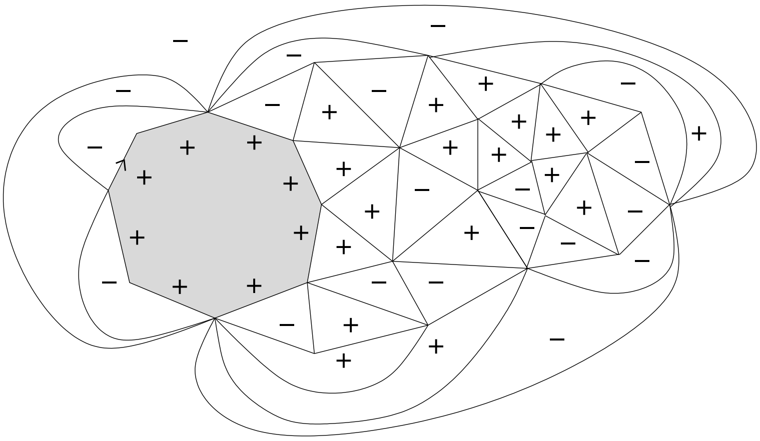


Unmarked faces are required to have degree at least 3. And for the moment, we consider that marked faces carry only spin $+$, and as usual, marked faces may have any degree, and must have a marked edge.

Definition 1.1 *The set $\mathbb{M}_k^{(g)}(v)$ is defined to be the set of connected oriented Ising maps of given genus g, with given number of marked faces k, and given number of vertices v, which are obtained by gluing those (oriented) pieces together.*

Like for uncolored maps, one easily proves, by computing the Euler characteristics, that this is a finite set.

Example of a typical map contributing to $\mathbb{M}_1^{(0)}$, it is a planar triangulation, with only one marked face of perimeter $l_1 = 8$:



We wish to enumerate those maps, recording numbers of all kinds of faces, and also recording the numbers of edges gluing faces of the same color $++$ or $--$, or of different colors $+-$.

Definition 1.2 *We define the generating function*

$$\begin{aligned} & W_k^{(g)}(x_1, \dots, x_k; t_3, \dots, t_d, \tilde{t}_3, \dots, \tilde{t}_{\tilde{d}}, c_{++}, c_{--}, c_{+-}; t) \\ = \; & \frac{t\, \delta_{k,1}\delta_{g,0}}{x_1} \\ & + \sum_{v=1}^{\infty} t^v \sum_{\Sigma \in \mathbb{M}_k^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\, t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}\, \tilde{t}_3^{\tilde{n}_3(\Sigma)}\, \tilde{t}_4^{\tilde{n}_4(\Sigma)} \dots \tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{x_1^{1+l_1(\Sigma)}\, x_2^{1+l_2(\Sigma)} \dots x_k^{1+l_k(\Sigma)}} \\ & \qquad \frac{1}{\#\mathrm{Aut}(\Sigma)}\, c_{++}^{n_{++}(\Sigma)}\, c_{--}^{n_{--}(\Sigma)}\, c_{+-}^{n_{+-}(\Sigma)} \end{aligned}$$

where $n_{ij}(\Sigma)$ is the number of edges separating two faces of colors i and j.

$W_k^{(g)}(x_1, \dots, x_k; t_3, \dots, t_d, \tilde{t}_3, \dots, \tilde{t}_{\tilde{d}}, c_{++}, c_{--}, c_{+-}; t)$ *is a formal power series in powers of t:*

$$W_k^{(g)} \in \mathbb{Q}[\{1/x_i\}, \{t_k\}, \{\tilde{t}_k\}, c_{++}, c_{--}, c_{+-}][[t]].$$

As usual, we write only the x_i dependence explicitely, and for short we shall write:

$$W_k^{(g)}(x_1, \dots, x_k; t_3, \dots, t_d, \tilde{t}_3, \dots, \tilde{t}_{\tilde{d}}, c_{++}, c_{--}, c_{+-}; t) = W_k^{(g)}(x_1, \dots, x_k)$$

and

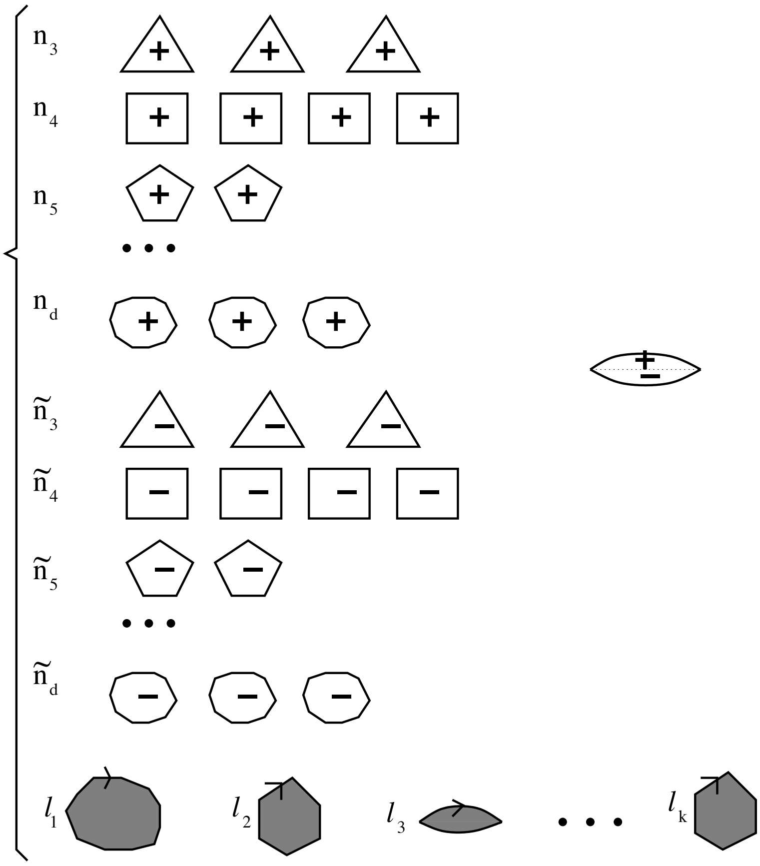
$$F_g = W_0^{(g)}.$$

Notice, that since a face can be glued to itself, the two faces on both sides of an edge, may be not distinct.

It is not so easy to write directly some Tutte-like equations for $W_k^{(g)}$, by removing the marked edge recursively on those maps. In fact, it is much easier to first consider a slightly different set of maps.

1.1 Reformulation

Instead of the previous Ising model, let us introduce another model. Consider maps, whose unmarked pieces can be of spin + or −, and also with some bicolored pieces of degree 2. We add the constraint that edges can be glued together along an edge only if the spin is the same on both sides of the edge.



Definition 1.3 *We define a generating function, with a weight c for that new piece, as well as a weight* $1/a$ *per number of* $++$ *edges, and* $1/b$ *per number of* $--$ *edges:*

$$
\begin{aligned}
& \hat{W}_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde{t}_3,\dots,\tilde{t}_{\tilde{d}},a,b,c;t) \\
= \quad & \frac{t\,\delta_{k,1}\delta_{g,0}}{x_1} \\
& +\sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\hat{\mathbb{M}}_k^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\,t_4^{n_4(\Sigma)}\,\dots\,t_d^{n_d(\Sigma)}\,\tilde{t}_3^{\tilde{n}_3(\Sigma)}\,\tilde{t}_4^{\tilde{n}_4(\Sigma)}\,\dots\,\tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{x_1^{1+l_1(\Sigma)}\,x_2^{1+l_2(\Sigma)}\,\dots\,x_k^{1+l_k(\Sigma)}}
\end{aligned}
$$

$$\frac{1}{\#\mathrm{Aut}(\Sigma)}\; a^{-n_{++}(\Sigma)}\, b^{-n_{--}(\Sigma)}\, c^{\hat{n}(\Sigma)}$$

where $\hat{n}(\Sigma)$ is the number of bicolored pieces.

In this definition, the coefficient of t^v is a formal power series in c (indeed $\hat{\mathbb{M}}_k^{(g)}(v)$ is not a finite set, because we can glue together as many bicolored pieces as we wish without changing the number of vertices, but for each power of c, there is a finite number of maps):

$$\hat{W}_k^{(g)} \in \mathbb{Q}[\{1/x_i\},\{t_k\},\{\tilde{t}_k\},1/a,1/b][[c]][[t]].$$

The reason why we have introduced this model, is because it coincides with the Ising model:

Theorem 1.1 *The generating function $\hat{W}_n^{(g)}$ of this model coincides with the generating function $W_n^{(g)}$ of the Ising model,*

$$\begin{aligned} & W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde{t}_3,\dots,\tilde{t}_{\tilde{d}},c_{++},c_{--},c_{+-};t) \\ & \quad = \hat{W}_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d,\tilde{t}_3,\dots,\tilde{t}_{\tilde{d}},a,b,c;t). \end{aligned}$$

with the identification:

$$\begin{pmatrix} c_{++} & c_{+-} \\ c_{+-} & c_{--} \end{pmatrix}^{-1} = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}.$$

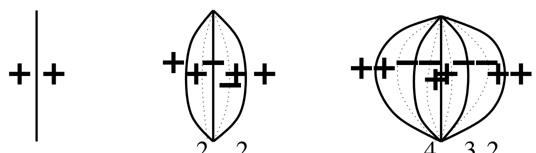
i.e.

$$c_{++} = \frac{b}{ab-c^2} \quad , \quad c_{--} = \frac{a}{ab-c^2} \quad , \quad c_{+-} = \frac{c}{ab-c^2}.$$

proof:

The sum over powers of c, is a geometrical series and can be performed explicitly.

We may glue several bicolored pieces so that both external sides have spin $+$:

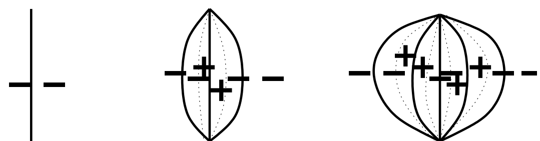


1/a c^2/a^2b c^4/a^3b^2

that corresponds to an effective $++$ edge gluing weight:

$$c_{++} = \frac{1}{a}\sum_k \frac{c^{2k}}{a^k\, b^k} = \frac{b}{ab-c^2}$$

Similarly we may glue such pieces so that both external sides have spin $-$:

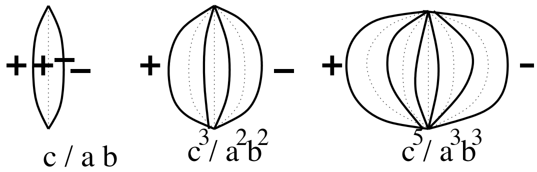


1/b c^2/b^2a c^4/b^3a^2

which corresponds to an effective $--$ edge gluing weight:

$$c_{--} = \frac{1}{b} \sum_k \frac{c^{2k}}{a^k\, b^k} = \frac{a}{ab - c^2}$$

And Similarly we may glue such pieces so that external sides have spin $+$ and $-$:



which corresponds to an effective $+-$ edge gluing weight:

$$c_{+-} = \frac{c}{ab} \sum_{k=0}^{\infty} \frac{c^{2k}}{a^k\, b^k} = \frac{c}{ab - c^2}.$$

Finally we recognize the matrix relationship

$$\begin{pmatrix} c_{++} & c_{+-} \\ c_{+-} & c_{--} \end{pmatrix}^{-1} = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}.$$

$\square$

2 Tutte-like equations

Definition 2.1 *We define the generating function of maps of genus g with n marked faces of given perimeters:*

$$\mathcal{T}^{(g)}_{l_1,\dots,l_n} = (-1)^n \operatorname{Res}\, x_1^{l_1} \dots x_n^{l_n}\, W_n^{(g)}(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

We have:

$$W_n^{(g)}(x_1, \dots, x_n) = \sum_{l_1,\dots,l_n} \frac{\mathcal{T}^{(g)}_{l_1,\dots,l_n}}{x_1^{l_1+1} \dots x_n^{l_n+1}}, \qquad \text{(VIII-2-1)}$$

as usual this equality is an equality of formal power series in t, and for each power of t, the sum over $l_1, \dots, l_n$ is a finite sum.

Definition 2.2 *Let us also define $\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n}$ to be the generating function of maps of genus g, and $n+1$ marked faces. n of the marked faces are usual marked faces carrying spin $+$, they have degrees l_i, $i = 1, \dots, n$, and one marked face, is of degree $l + k$, so that there are l consecutive edges gluing to spin $+$, and k consecutive edges gluing to spin $-$. If $k \geq 1$, the marked edge on that marked face can always be assumed to be the first $+$ edge on the right side of a $-$ edge.*

Similarly, we define $G^{(g)}_{n,k}(x;x_1,\dots,x_n)$ to be the generating series where we sum over perimeter of marked faces weighted by $x_i^{-l_i-1}$ and x^{-l-1}. We have:

$$\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n} = (-1)^{n+1}\ \mathrm{Res}\ \ x^l\ x_1^{l_1}\dots x_n^{l_n}\ G^{(g)}_{n,k}(x;x_1,\dots,x_n)\,dx\,dx_1\dots dx_n$$

(notice that we don't sum over k), and

$$G^{(g)}_{n,k}(x;x_1,\dots,x_n) = \sum_{l,l_1,\dots,l_n} \frac{\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\dots,l_n}}{x^{l+1}\ \ x_1^{l_1+1}\dots x_n^{l_n+1}}. \qquad \text{(VIII-2-2)}$$

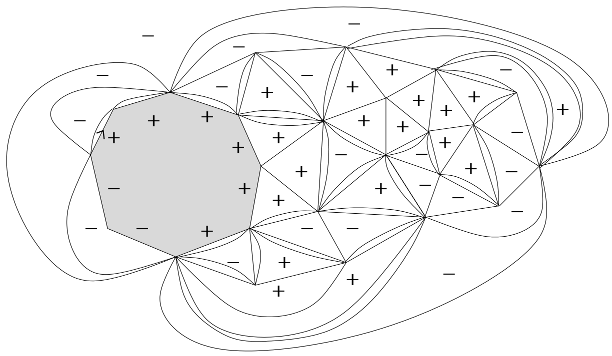
Also, if $k=0$, we recover:

$$\hat{\mathcal{T}}^{(g)}_{l,0;l_1,\dots,l_n} = \mathcal{T}^{(g)}_{l,l_1,\dots,l_n}$$

and

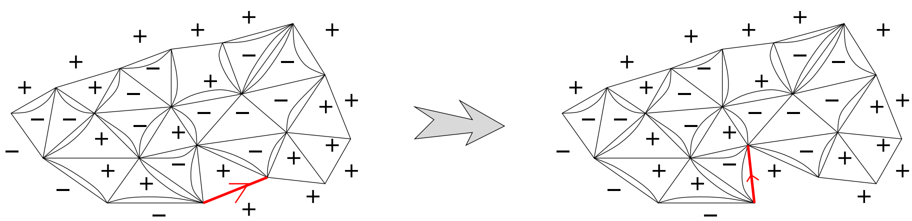
$$G^{(g)}_{n,0}(x;x_1,\dots,x_n) = W^{(g)}_{n+1}(x,x_1,\dots,x_n).$$

For example, here is a typical map contributing to $\hat{\mathcal{T}}^{(0)}_{6,2}$:

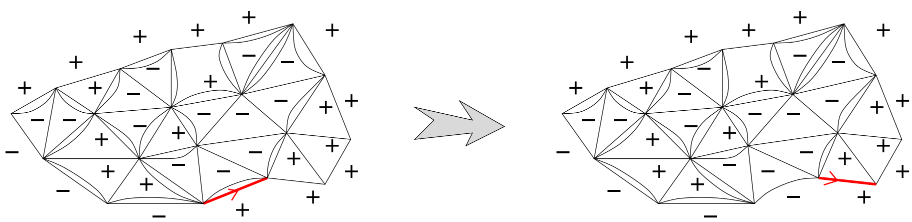


Consider a map contributing to $\hat{\mathcal{T}}^{(g)}_{l+1,k;l_1,\dots,l_n}$. On the other side of the marked edge (which is a + edge), there can be either:

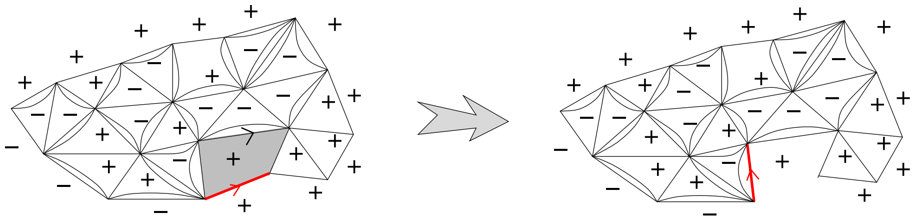
• an unmarked spin + face of degree j, with $3\le j\le d$, and removing the marked edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l+j-1,k;l_1,\dots,l_n}$ weighted by t_j/a.



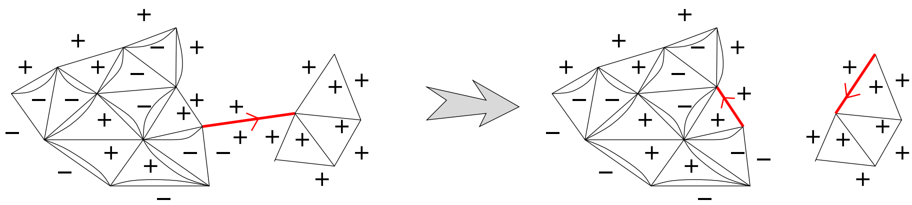
• a bicolored face (+−), and removing the marked edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l,k+1;l_1,\dots,l_n}$ weighted by c/a.



• the i^{th} marked face of degree l_i, and removing the marked edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l+l_i-1,k;l_1,\dots,\cancel{l_i},\dots,l_n}$ weighted by l_i/a.



• the same marked face. In that case, removing the marked edge either disconnects the map into two maps, or if there was a handle relating the 2 sides, it gives a map of lower genus $g-1$ and one more boundary.

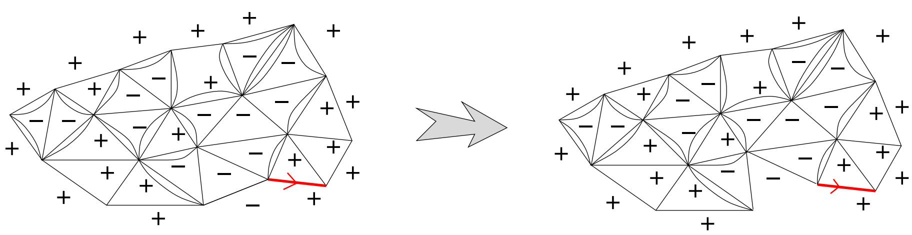


Finally, we see that bijectively removing the marked edge implies the following relationships among generating functions:

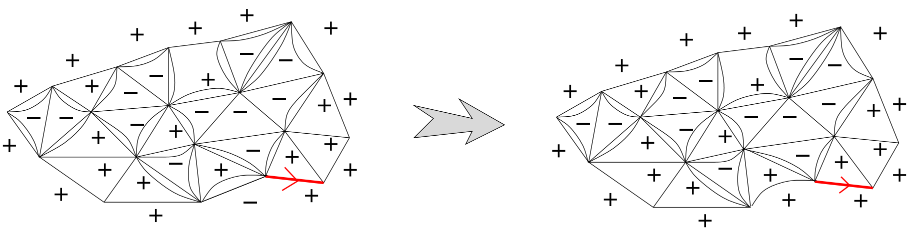
$$\begin{aligned} a\,\hat{\mathcal{T}}^{(g)}_{l+1,k;l_1,\dots,l_n} &= \sum_{j=3}^{d} t_j \hat{\mathcal{T}}^{(g)}_{l+j-1,k;l_1,\dots,l_n} \\ &\quad + c\,\hat{\mathcal{T}}^{(g)}_{l,k+1;l_1,\dots,l_n} \\ &\quad + \sum_{i=1}^{n} l_i\, \hat{\mathcal{T}}^{(g)}_{l+l_i-1,k;l_1,\dots,\cancel{l_i},\dots,l_n} \\ &\quad + \sum_{j=1}^{l-1} \hat{\mathcal{T}}^{(g-1)}_{j,k;l-j-1,l_1,\dots,l_n} \\ &\quad + \sum_{j=1}^{l-1} \sum_{h=0}^{g} \sum_{J\subset\{l_1,\dots,l_n\}} \hat{\mathcal{T}}^{(h)}_{j,k;I} \mathcal{T}^{(g-h)}_{l-j-1,J\setminus I} . \end{aligned}$$

Since those equations may increase k by 1, they can't be closed, and thus we need another equation. For that purpose, consider a map contributing to $\hat{\mathcal{T}}^{(g)}_{l,1;l_1,\dots,l_n}$ with $k=1$. It has a unique $-$ edge, and on the other side of the $-$ edge, there can be either:

• an unmarked spin $-$ face of degree j, with $3\leq j\leq \tilde{d}$, and removing the $-$ edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l,j-1;l_1,\dots,l_n}$ weighted by $\tilde{t}_j/b$.



• a bicolored face $(+-)$, and removing the $-$ edge gives a map of $\hat{\mathcal{T}}^{(g)}_{l+1,0;l_1,\ldots,l_n}$ weighted by c/b.



There cannot be another marked face, neither the same marked face, because there is no other $-$ edge to glue to.

Finally, in terms of generating functions we have

$$b\,\hat{\mathcal{T}}^{(g)}_{l,1;l_1,\ldots,l_n} = \sum_{j=3}^{\tilde{d}} \tilde{t}_j \hat{\mathcal{T}}^{(g)}_{l,j-1;l_1,\ldots,l_n} + c\,\hat{\mathcal{T}}^{(g)}_{l+1,0;l_1,\ldots,l_n}.$$

2.1 Equation for generating functions

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It is more interesting to define the Following series:

$$V_1'(x) = ax - \sum_{j=3}^{d} t_j\, x^{j-1} \qquad ,\ t_2 = -a$$

$$V_2'(y) = by - \sum_{j=3}^{\tilde{d}} \tilde{t}_j\, y^{j-1} \qquad ,\ \tilde{t}_2 = -b$$

$$c\,Y(x) = V_1'(x) - W_1^{(0)}(x).$$

And:

$$U_n^{(g)}(x,y;x_1,\ldots,x_n) = (-cV_2'(y) + c^2 x)\delta_{n,0}\delta_{g,0} - \sum_{j=2}^{\tilde{d}} \sum_{k=0}^{j-2} \tilde{t}_j\, y^{j-2-k}\ G^{(g)}_{n,k}(x;x_1,\ldots,x_n) \tag{VIII-2-3}$$

$$P_n^{(g)}(x,y;x_1,\ldots,x_n) = \sum_{j=2}^{d} \sum_{\tilde{j}=2}^{\tilde{d}} \sum_{l=0}^{j-2} \sum_{k=0}^{\tilde{j}-2} \sum_{l_1,\ldots,l_n} t_j\, \tilde{t}_{\tilde{j}}\, x^{j-2-l}\, y^{\tilde{j}-2-k}\ \frac{\hat{\mathcal{T}}^{(g)}_{l,k;l_1,\ldots,l_n}}{x_1^{l_1+1}\ldots x_n^{l_n+1}}. \tag{VIII-2-4}$$

Notice that $U_n^{(g)}(x,y;x_1,\ldots,x_n)$ is a polynomial in y, and $P_n^{(g)}(x,y;x_1,\ldots,x_n)$ is a polynomial in both x and y.

In terms of these, the loop equations become:

Theorem 2.1 *The generating functions of the Ising model satisfy the following set of equations (called "loop equations" or "Tutte–like equations"):*

$$c\,(y - Y(x))\,U_n^{(g)}(x,y;x_1,\ldots,x_n) + W^{(g)}_{n+1}(x,x_1,\ldots,x_n)\,U_0^{(0)}(x,y)$$

$$
\begin{aligned}
& +U^{(g-1)}_{n+1}(x,y;x,x_1,\ldots,x_n) \\
& +\sum_{h=0}^{g}\sum_{I\subset\{x_1,\ldots,x_n\}}' U^{(h)}_{\#I}(x,y;I)\,W^{(g-h)}_{n+1-\#I}(x,J\setminus I) \\
& +\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{U^{(g)}_n(x,y;\{x_1,\ldots,x_n\}\setminus\{x_i\})-U^{(g)}_n(x_i,y;\{x_1,\ldots,x_n\}\setminus\{x_i\})}{x-x_i} \\
= \; & c\,((V_1'(x)-cy)(V_2'(y)-cx)+t\,c)\,\delta_{n,0}\delta_{g,0}-P^{(g)}_n(x,y;x_1,\ldots,x_n). \qquad \text{(VIII-2-5)}
\end{aligned}
$$

3 Solution of loop equations

Loop equation eq.(VIII-2-5) look substantially more intricated than Tutte's equations for maps without Ising spins, however, as we shall see, the symplectic invariants of chapter VII still give the solution.

3.1 The disc: spectral curve

The disc corresponds to $n=0$ and $g=0$, for which the loop equation reads

$$c\,(y-Y(x))\;U^{(0)}_0(x,y)=c(V_1'(x)-cy)(V_2'(y)-cx)-P^{(0)}_0(x,y)+t\,c^2.$$

The right hand side is a polynomial of both x and y, and we call it $E(x,y)$:

$$E(x,y)=(V_1'(x)-cy)(V_2'(y)-cx)-\frac{1}{c}\,P^{(0)}_0(x,y)+t\,c.$$

Notice that $(V_1'(x)-cy)(V_2'(y)-cx)$ is a polynomial of x of degree d and of y of degree $\tilde d$, whereas $P^{(0)}_0(x,y)$ is a polynomial of x of degree $d-2$ and of y of degree $\tilde d-2$.

The loop equation for the disc can thus be written

$$(y-Y(x))\;U^{(0)}_0(x,y)=E(x,y). \qquad \text{(VIII-3-1)}$$

Since $U^{(0)}_0(x,y)$ is a polynomial in y, it has no pole at $y=Y(x)$, and thus, by substituting $y\to Y(x)$ we get

$$E(x,Y(x))=0$$

This equation shows that $Y(x)$ is an algebraic function of x, and therefore $W^{(0)}_1(x)=V_1'(x)-cY(x)$ is an algebraic function of x. Moreover, we leave to the reader a straightforward generalization of the 1-cut Brown's Lemma (see section 1.2 in chapter III) , which shows that this algebraic equation must be of genus 0, and thus the solution can be parametrized by rational functions. Like Zhukowski variable, we define:

$$\begin{cases} x(z)=\gamma\,z+\sum_{k=0}^{\tilde d-1}\alpha_k z^{-k} \\ Y(x(z))=y(z)=\gamma\,z^{-1}+\sum_{k=0}^{d-1}\beta_k z^k \end{cases}$$

Writing that this parametrization is solution of $E(x(z),y(z))=0$, determines all the coefficients γ,α_k,β_k, as well as all coefficients of the polynomial $P^{(0)}_0(x,y)$, as algebraic functions of t, a, b, c, t_j, $\tilde t_j$, (they are thus algebraic power series in t).

Theorem 3.1 *The disc amplitude $Y(x) = \frac{1}{c}(V_1'(x) - W_1^{(0)}(x))$ is determined as follows: Let*

$$\left\{\begin{array}{l} x(z) = \gamma\, z + \sum_{k=0}^{\tilde d-1} \alpha_k z^{-k} \\ Y(x(z)) = y(z) = \gamma\, z^{-1} + \sum_{k=0}^{d-1} \beta_k z^k \end{array}\right.$$

where $\gamma, \alpha_k, \beta_k$ are the unique solutions of the system of equations

$$V_1'(x(z)) - cy(z) \underset{z\to\infty}{\sim} \frac{t}{\gamma z} + O(1/z^2)$$

$$V_2'(y(z)) - cx(z) \underset{z\to 0}{\sim} \frac{t\, z}{\gamma} + O(z^2)$$

such that

$$\gamma^2 = \frac{ct}{ab - c^2} + O(t^2).$$

Then the disc amplitude $Y(x)$ is:

$$Y(x(z)) = y(z).$$

proof:

First, notice that there is a unique solution of loop equations which is a power series in t (other solutions would have negative or fractional powers of t), indeed, those eqations correspond to recursively adding edges, and they uniquely allow to construct every Ising map). One can check that this parametric curve is indeed solution of loop equations, and it is a power series of t, therefore it is the solution needed. $\square$

Definition 3.1 *The spectral curve $\mathcal{E} = (x, y)$ is the pair of rational functions $x(z)$ and $y(z)$:*

$$\left\{\begin{array}{l} x(z) = \gamma z + \sum_{k=0}^{\tilde d-1} \alpha_k z^{-k} \\ y(z) = \frac{\gamma}{z} + \sum_{k=0}^{d-1} \beta_k z^k \end{array}\right.$$

where $\alpha_k, \beta_k, \gamma$ are determined by

$$V_1'(x(z)) = cy(z) + \frac{t}{\gamma z} + O(z^{-2}) \qquad , \qquad V_2'(y(z)) = cx(z) + \frac{t\, z}{\gamma} + O(z^2)$$

and we choose the unique solution such that $\gamma^2 = O(t)$ at small t.

Theorem 3.2 (Algebraic equation) *The functions $x(z)$ and $y(z)$ are related by the algebraic equation $E(x(z), y(z)) = 0$, where $E(x, y)$ is the polynomial of two variables given by the resultant:*

$$E(x,y) = \frac{-\,(-1)^{\tilde d}\, c^2}{\gamma^{d+\tilde d-2}} \det \begin{pmatrix}
\gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{d_2} & & & & \\
 & \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{d_2} & & & \\
 & & \ddots & & & & \ddots & & \\
 & & & \ddots & & & & \ddots & \\
 & & & & \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{d_2} \\
\beta_{d_1} & \dots & & \beta_1 & \beta_0 - y & \gamma & & & \\
 & \beta_{d_1} & \dots & & \beta_1 & \beta_0 - y & \gamma & & \\
 & & \ddots & & & & & \ddots & \\
 & & & \beta_{d_1} & \dots & & \beta_1 & \beta_0 - y & \gamma
\end{pmatrix}$$

proof:

The resultant vanishes if and only if there is a z which is a common zero of $x(z) - x = 0$ and $y(z) - y = 0$, indeed observe that the vector $(1, z^{-1}, z^{-2}, z^{-3}, \ldots)$ is an eigenvector of that matrix, for the eigenvalue 0. The resultant is thus a polynomial of x and y of the correct degree, which vanishes exactly when $E(x, y)$ vanishes, it is thus proportional to $E(x, y)$. The prefactor is determined by matching the large x behavior of $E(x, y) \sim -cxV_1'(x)$. $\square$

Theorem 3.3 Variational principle

One can determine the coefficients $\alpha_k, \beta_k, \gamma$ by extremizing the functional:

$$\mu(\gamma, \alpha_k, \beta_k) = \mathop{\mathrm{Res}}_{z\to\infty} \left(V_1(x(z)) + V_2(y(z)) - c\,x(z)y(z)\right) \frac{dz}{z} + 2t\,\ln\gamma$$

proof:

$$\begin{aligned}
\frac{\partial\mu}{\partial\alpha_k} &= \mathop{\mathrm{Res}}_{z\to\infty} \left(V_1'(x(z)) - cy(z)\right) \frac{dz}{z^{k+1}} \\
&= \mathop{\mathrm{Res}}_{z\to\infty} \left(\frac{t}{\gamma z} + O(1/z^2)\right) \frac{dz}{z^{k+1}} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\mu}{\partial\beta_k} &= \mathop{\mathrm{Res}}_{z\to\infty} \left(V_2'(y(z)) - cxz)\right) z^{k-1}\,dz \\
&= -\mathop{\mathrm{Res}}_{z\to 0} \left(V_2'(y(z)) - cxz)\right) z^{k-1}\,dz \\
&= -\mathop{\mathrm{Res}}_{z\to 0} \left(\frac{tz}{\gamma} + O(z^2)\right) z^{k-1}\,dz \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\mu}{\partial\gamma} &= \mathop{\mathrm{Res}}_{z\to\infty} \left(V_1'(x(z)) - cy(z)\right) dz + \mathop{\mathrm{Res}}_{z\to\infty} \left(V_2'(y(z)) - cx(z)\right) \frac{dz}{z^2} + \frac{2t}{\gamma} \\
&= \mathop{\mathrm{Res}}_{z\to\infty} \left(V_1'(x(z)) - cy(z)\right) dz - \mathop{\mathrm{Res}}_{z\to 0} \left(V_2'(y(z)) - cx(z)\right) \frac{dz}{z^2} + \frac{2t}{\gamma} \\
&= \mathop{\mathrm{Res}}_{z\to\infty} \left(\frac{t}{\gamma z} + O(1/z^2)\right) dz - \mathop{\mathrm{Res}}_{z\to 0} \left(\frac{tz}{\gamma} + O(z^2)\right) \frac{dz}{z^2} + \frac{2t}{\gamma} \\
&= -\frac{t}{\gamma} - \frac{t}{\gamma} + \frac{2t}{\gamma} \\
&= 0
\end{aligned}$$

Reciprocally, if $\partial\mu/\partial\alpha_k = 0$ for all k that means

$$\mathop{\mathrm{Res}}_{z\to\infty} \left(V_1'(x(z)) - cy(z)\right) \frac{dz}{z^{k+1}} = 0$$

and thus $V_1'(x(z)) - cy(z) = O(1/z)$. Similarly, $\partial\mu/\partial\beta_k = 0$ for all k implies that $V_2'(y(z)) - cx(z) = O(z)$. And then, $\partial\mu/\partial\gamma = 0$ implies that $V_1'(x(z)) - cy(z) \sim t/x(z)$ and $V_2'(y(z)) - cx(z) \sim t/y(z)$.

$\square$

3.2 Example: Ising model on quadrangulations

We chose only t_4 and $\tilde{t}_4$ non-vanishing.

We have $V_1'(x) = ax - t_4 x^3$ and $V_2'(y) = by - \tilde{t}_4 y^3$.

Theorem 3.1 says that we should look for 2 rational functions $x(z)$ and $y(z)$ of the form (we exploit the parity of V_1 and V_2):

$$x(z) = \gamma z + \alpha_1 z^{-1} + \alpha_3 z^{-3} \qquad , \qquad y(z) = \gamma/z + \beta_1 z + \beta_3 z^3.$$

We need to compute:

$$V_1'(x(z)) = a(\gamma z + \alpha_1 z^{-1}) - t_4(\gamma^3 z^3 + 3\alpha_1\gamma^2 z + 3\alpha_3\gamma^2 z^{-1} + 3\alpha_1^2\gamma z^{-1}) + O(z^{-3})$$

and thus:

$$c\,\beta_3 = -t_4\gamma^3 \qquad , \qquad c\beta_1 = a\gamma - 3t_3\alpha_1\gamma^2 \qquad , \qquad a\alpha_1 - 3t_4(\alpha_3\gamma^2 + \alpha_1^2\gamma) - c\gamma = \frac{t}{\gamma},$$

and similarly by computing $V_2'(y(z)) - cx(z)$:

$$c\,\alpha_3 = -\tilde{t}_4\gamma^3 \qquad , \qquad c\alpha_1 = b\gamma - 3\tilde{t}_3\beta_1\gamma^2 \qquad , \qquad b\beta_1 - 3t_4(\beta_3\gamma^2 + \beta_1^2\gamma) - c\gamma = \frac{t}{\gamma}.$$

Let us consider for simplicity the symmetric case, where $a = b$ and $t_4 = \tilde{t}_4$. In that case we shall find $\alpha_i = \beta_i$, and thus:

$$c\,\alpha_3 = -t_4\gamma^3 \qquad , \qquad c\alpha_1 = a\gamma - 3t_4\alpha_1\gamma^2 \qquad , \qquad a\alpha_1 - 3t_4(\alpha_3\gamma^2 + \alpha_1^2\gamma) - c\gamma = \frac{t}{\gamma},$$

That gives an algebraic equation of degree 5 for γ^2:

$$(c + 3t_4\gamma^2)^2\,(t + c\gamma^2 - 3\frac{t_4^2}{c}\,\gamma^6) - ca^2\gamma^2 = 0.$$

and we chose the unique solution which behaves at small t like

$$\gamma^2 = \frac{c\,t}{a^2 - c^2} + O(t^2).$$

We then have

$$\alpha_1 = \frac{a\,\gamma}{c + 3t_4\gamma^2} \qquad , \qquad \alpha_3 = -\,\frac{t_4\gamma^3}{c}.$$

3.3 All topologies generating functions

Then, knowing this spectral curve we have for Ising maps the equivalent of theorem III.3.1 :

Theorem 3.4 *The generating functions counting Ising maps, are given by the symplectic invariants of chapter VII:*

$$F_g = \mathcal{F}_g(\mathcal{E})$$

For the spectral curve $\mathcal{E} = (\mathbb{C}, x, y, B(z,z') = dz\,dz'/(z-z')^2)$. The $\omega_n^{(g)}(\mathcal{E})$'s of chapter VII give the generating functions of maps with n marked faces of spin $+$:

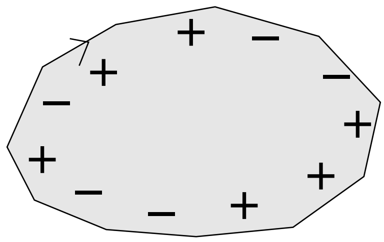
$$W_n^{(g)}(x(z_1),\dots,x(z_n))\,dx(z_1)\dots dx(z_n) = \omega_n^{(g)}(z_1,\dots,z_n) + \delta_{n,1}\,\delta_{g,0}\,V_1'(x(z_1))\,dx(z_1) + \delta_{n,2}\,\delta_{g,0}\,\frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2} \qquad (VIII-3-2)$$

This theorem was proved in [], and in this version of the book, we skip the proof. We just mention that the proof is much more technical than for uncolored maps, it is not at all a straightforward extension of chapter III.

4 Mixed boundary conditions

So far, we have been considering marked faces, as well as unmarked faces carrying one spin in their center.

Now, let us also consider marked faces having different spins on their sides (unmarked faces will always have only one spin, either $+$ or $-$). A typical marked face can be:



Let us construct a good set of generating functions for counting maps with such marked faces with spin boundary conditions.

4.1 Maps with mixed boundaries

First, consider maps having n marked faces, of respective perimeters $l_1,\dots,l_n$, such that the i^{th} marked face has $2k_i$ changes of boundary conditions:

$$\text{marked face } i = l_{i,1}\,\text{spin}+, \tilde{l}_{i,1}\,\text{spin}-, l_{i,2}\,\text{spin}+, \tilde{l}_{i,2}\,\text{spin}-, \dots, l_{i,k_i}\,\text{spin}+, \tilde{l}_{i,k_i}\,\text{spin}-.$$

$$l_i = \sum_{j=1}^{k_i} l_{i,j} + \tilde{l}_{i,j}.$$

Our goal is to compute the generating function which enumerates such configurations:

$$\sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\mathbb{M}^{(g)}_{n;k_1,\dots,k_n}(v)} \frac{t_3^{n_3(\Sigma)}\,t_4^{n_4(\Sigma)}\,\dots\,t_d^{n_d(\Sigma)}\;\tilde{t}_3^{\tilde{n}_3(\Sigma)}\,\tilde{t}_4^{\tilde{n}_4(\Sigma)}\,\dots\,\tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{\prod_{i=1}^n\prod_{j=1}^{k_i} x_{i,j}^{1+l_{i,j}(\Sigma)}\,y_{i,j}^{1+\tilde{l}_{i,j}(\Sigma)}}$$

$$\frac{1}{\#\mathrm{Aut}(\Sigma)}\ a^{-n_{++}(\Sigma)}\, b^{-n_{--}(\Sigma)}\, c^{\hat{n}(\Sigma)}$$

This generating function depends on k parameters of type $x_{i,j}$ (associated to spin $+$ boundaries of length $l_{i,j}$) and k parameters of type $y_{i,j}$ (associated to spin $-$ boundaries of length $\tilde{l}_{i,j}$), where $2k$ is the total number of boundary condition changes:

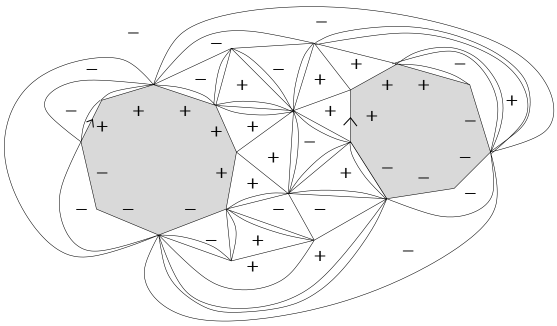
$$k = \sum_{i=1}^{n} k_i.$$

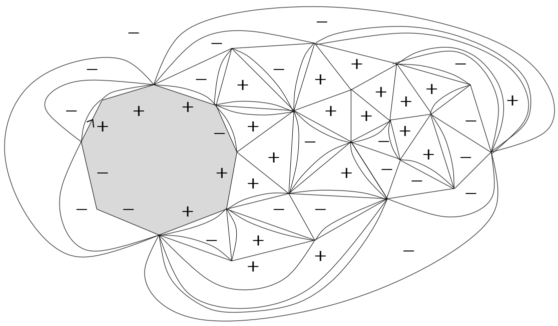
Fixed k

From now on, it will be better to compute at once all generating functions with a given k, i.e. with an arbitrary number of marked faces, provided that the total number of boundary condition changes is $2k$.

For instance for $k = 2$, we have either 2 marked faces with $k_1 = k_2 = 1$, or 1 marked face with $k_1 = 2$.

For example the following maps both correspond to $k = 2$. The first one has two marked faces, one with $l_{1,1} = 5, \tilde{l}_{1,1} = 3$ and one with $l_{2,1} = 3, \tilde{l}_{2,1} = 4$, and the second map has only one marked face with $l_{1,1} = 3, \tilde{l}_{1,1} = 1, l_{1,2} = 2, \tilde{l}_{1,2} = 2$:

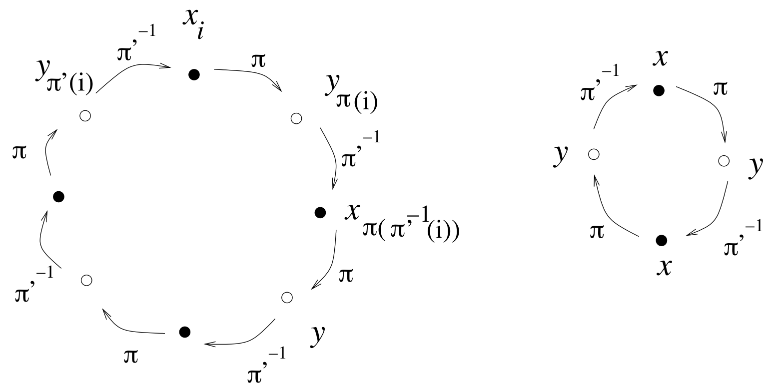




For that purpose, let us consider k parameters x_i, $i = 1, \ldots, k$ associated to spin $+$ pieces of boundaries of length l_i, and k parameters y_i, $i = 1, \ldots, k$ associated to spin $-$ pieces of boundaries of length $\tilde{l}_i$. Let us consider all possible boundary conditions which can be encoded by those $2k$ parameters.

Consider x_1, it is associated to a piece of $+$ boundary of length l_1 of some marked face. Going around the marked face (in the direction defined by the map orientation), it must be followed by a $-$ piece of boundary $y_{\pi(1)}$ of length $\tilde{l}_{\pi(1)}$, where π is some permutation of indices. Then, the $-$ piece of boundary $y_{\pi(1)}$ must be followed by a $+$ piece of boundary, let us call it $x_{\pi'^{-1}(\pi(1))}$, where π' is another permutation. We proceed until we have completed a cycle around a marked face, i.e. until we have completed a cycle of the permutation $\pi'^{-1} \circ \pi$. Then we repeat the same procedure for all cycles of

$\pi'^{-1} \circ \pi$.



Considering all pairs of permutations (π, π') exhausts all possible boundary conditions with $2k$ changes of boundary spins.

Definition 4.1 *Let us define the following generating function:*

$$\begin{aligned} & \widehat{H}^{(g)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) \\ = \; & -c\,\delta_{g,0}\,\delta_{k,1} \\ & +\sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\mathbb{M}^{(g)}_{\pi,\pi'}(v)} \frac{t_3^{n_3(\Sigma)}\,t_4^{n_4(\Sigma)}\,\dots\,t_d^{n_d(\Sigma)}\,\tilde{t}_3^{\tilde{n}_3(\Sigma)}\,\tilde{t}_4^{\tilde{n}_4(\Sigma)}\,\dots\,\tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(\Sigma)}}{\prod_{i=1}^k x_i^{1+l_i(\Sigma)}\,y_i^{1+\tilde{l}_i(\Sigma)}} \\ & \qquad \frac{1}{\#\mathrm{Aut}(\Sigma)}\;a^{-n_{++}(\Sigma)}\,b^{-n_{--}(\Sigma)}\,c^{\hat{n}(\Sigma)} \end{aligned}$$

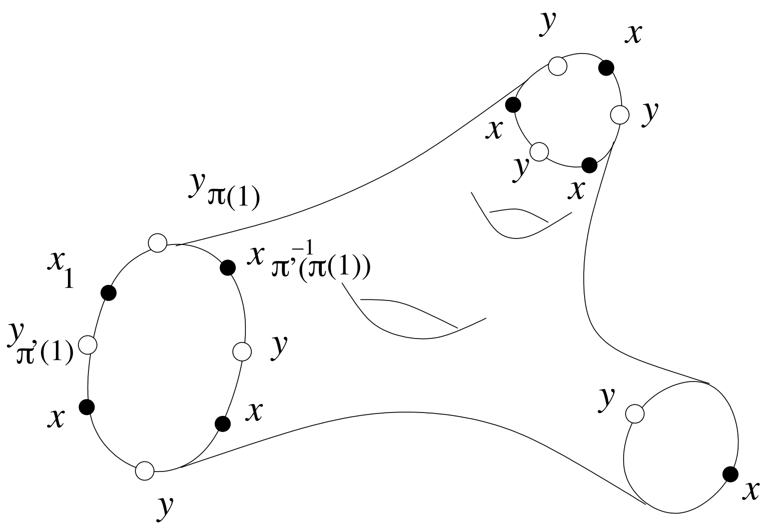
We also define the generating functions resummed with the genus:

$$\widehat{H}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)=\sum_{g=0}^{\infty}(N/t)^{2-2g-\ell(\pi'^{-1}\circ\pi)}\;\widehat{H}^{(g)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) \tag{VIII-4-1}$$

where $\ell(\pi'^{-1}\circ\pi)$ is the number of cycles of $\pi'^{-1}\circ\pi$. As usual, this equality is to be understood as an equality of formal series in t, and for each power of t, the sum over g is finite.

$\widehat{H}^{(g)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)$ *counts maps drawn on surfaces of genus g, with $\ell(\pi'^{-1}\circ\pi)$ boundaries, and whose boundaries are labeled by a sequence of x and y variables according to the cycles of $\pi'^{-1}\circ\pi$.*

Example of a surface of genus 2, with $k = 8$, and such that $\pi \circ \pi'^{-1}$ has 3 cycles:



Non connected generating functions

The formulae that will follow are better written using generating function for non-necessarily connected maps. But we require that connected pieces contain at least one boundary.

We define generating functions of non-connected maps, as the product of connected ones.

For example for $k = 1$ there is only one boundary, and the map must be connected, we define

$$H_{\mathrm{Id}_1,\mathrm{Id}_1}(x;y) = \widehat{H}_{\mathrm{Id}_1,\mathrm{Id}_1}(x;y).$$

For $k = 2$, if $(\pi, \pi') = (\mathrm{Id}_2, \mathrm{Id}_2)$, we see that $\pi \circ \pi'^{-1}$ has two cycles, so the maps can either be connected with 2 boundaries, or disconnected, thus we define:

$$H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1, x_2; y_1, y_2) = \widehat{H}_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1, x_2; y_1, y_2) + \widehat{H}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1; y_1)\, \widehat{H}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2; y_2)$$

and if $(\pi, \pi') = (\mathrm{Id}_2, (1, 2))$, we see that $\pi \circ \pi'^{-1}$ has only one cycle, so it must be connected and thus we define

$$H_{\mathrm{Id}_2,(1,2)}(x_1, x_2; y_1, y_2) = \widehat{H}_{\mathrm{Id}_2,(1,2)}(x_1, x_2; y_1, y_2).$$

And so on.

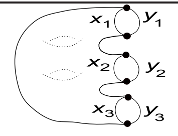
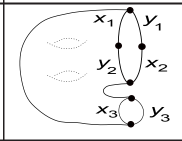
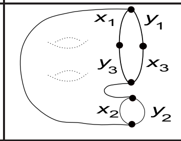
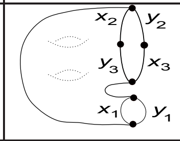
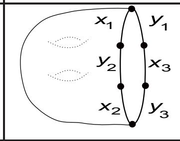
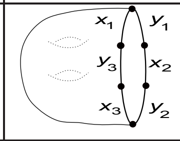
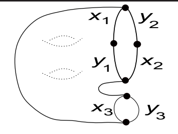
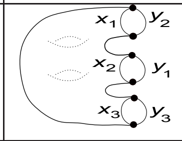
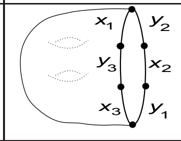
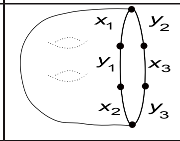
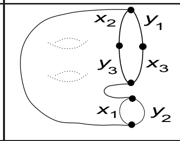
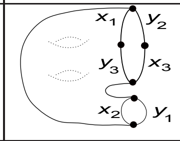
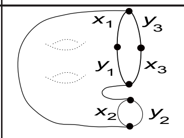
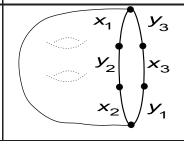
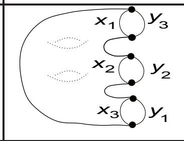
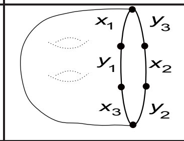
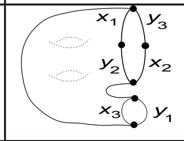
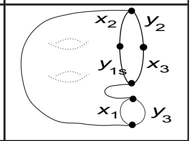
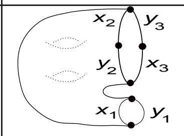
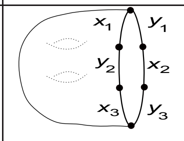
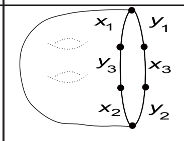
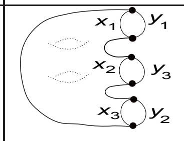
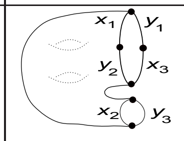
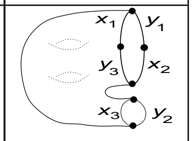
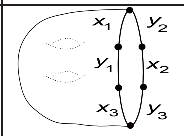
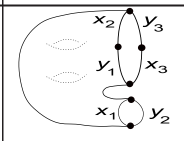
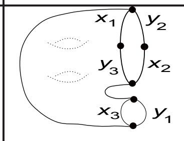
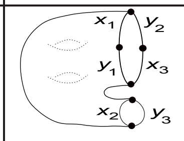
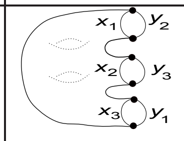
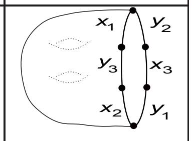
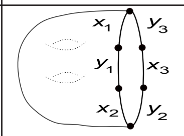
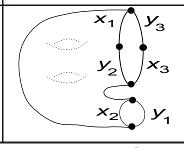
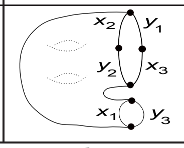
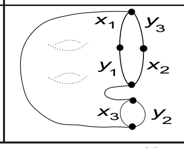
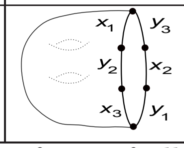
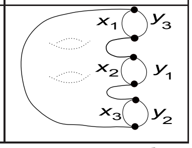
In general,

Definition 4.2 *$H^{(g)}_{\pi,\pi'}$ is defined as the sum of products of $\widehat{H}^{(g)}_{\pi_i,\pi'_i}$ for all possible ways of decomposing the permutation $\pi \circ \pi'^{-1}$ into a product disjoint permutations $\prod_i \pi_i \circ \pi_i'^{-1}$.*

The matrix generating function

For every pair of permutations of k variables (π, π'), we have defined a generating function $H_{\pi,\pi'}$. Let us now consider them all together into a $k! \times k!$ matrix.

Example with $k = 3$, we have the 6×6 matrix

	Id	(12)	(13)	(23)	(123)	(132)
Id						
(12)						
(13)						
(23)						
(123)						
(132)						

(here the pictures means that we take into account all surfaces of all genus, and possibly disconnected, with the corresponding boundaries).

We have several important results:

Theorem 4.1 *The matrix $H(x_1, \ldots, x_k; y_1, \ldots, y_k)$ is symmetric:*

$$H_{\pi,\pi'}(x_1, \ldots, x_k; y_1, \ldots, y_k) = H_{\pi',\pi}(x_1, \ldots, x_k; y_1, \ldots, y_k) \qquad \text{(VIII-4-2)}$$

proof:

It just consists in remarking that reversing the orientation of a map, gives another map, with the same number k of boundary conditions, and reversing the boundary just exchanges π and π'. $\square$

Commutation relations

Then, we have a non-trivial result:

Theorem 4.2 *We have*

$$[H(x_1, \ldots, x_k; y_1, \ldots, y_k), \mathcal{A}] = 0$$

$$\forall\, i=1,\dots,k\,,\qquad [H(x_1,\dots,x_k;y_1,\dots,y_k),\mathcal{A}_i]=0$$

where

$$(\mathcal{A}_i)_{\pi,\pi'}=\left\{\begin{array}{ll} y_{\pi(i)} & \text{if } \pi'=\pi \\ \frac{-t}{N\,c}\,\frac{1}{x_i-x_j} & \text{if } \pi'=\pi\circ(ij) \\ 0 & \text{otherwise}\end{array}\right. .$$

and $\mathcal{A}=\sum_i x_i\mathcal{A}_i$*:*

$$\mathcal{A}_{\pi,\pi'}=\left\{\begin{array}{ll} \sum_i x_i y_{\pi(i)} & \text{if } \pi'=\pi \\ \frac{-t}{N\,c} & \text{if } \pi^{-1}\circ\pi'=\text{transposition} \\ 0 & \text{otherwise}\end{array}\right. .$$

Example with $k=3$:

$$\mathcal{A}_1=\begin{pmatrix}
y_1 & \frac{-t}{N\,c}\frac{1}{x_1-x_2} & \frac{-t}{N\,c}\frac{1}{x_1-x_3} & 0 & 0 & 0 \\
\frac{-t}{N\,c}\frac{1}{x_1-x_2} & y_2 & 0 & 0 & 0 & \frac{-t}{N\,c}\frac{1}{x_1-x_3} \\
\frac{-t}{N\,c}\frac{1}{x_1-x_3} & 0 & y_3 & 0 & \frac{-t}{N\,c}\frac{1}{x_1-x_2} & 0 \\
0 & 0 & 0 & y_1 & \frac{-t}{N\,c}\frac{1}{x_1-x_3} & \frac{-t}{N\,c}\frac{1}{x_1-x_2} \\
0 & 0 & \frac{-t}{N\,c}\frac{1}{x_1-x_2} & \frac{-t}{N\,c}\frac{1}{x_1-x_3} & y_2 & 0 \\
0 & \frac{-t}{N\,c}\frac{1}{x_1-x_3} & 0 & \frac{-t}{N\,c}\frac{1}{x_1-x_2} & 0 & y_3
\end{pmatrix}$$

And we have the corollary

Theorem 4.3 *For every* ξ,η *complex numbers, the matrix* $H(x_1,\dots,x_k;y_1,\dots,y_k)$ *commutes with the matrix* $\mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta)$ *defined by:*

$$\mathcal{M}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta)=\prod_{i=1}^{k}\left(\delta_{\pi(i),\pi'(i)}-\frac{t}{N\,c}\frac{1}{(x_i-\xi)\,(y_{\pi(i)}-\eta)}\right). \tag{VIII-4-3}$$

(it is a symmetric matrix).

We have

$$\forall\,\xi,\,\eta\qquad,\qquad [H(x_1,\dots,x_k;y_1,\dots,y_k),\mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta)]=0 \tag{VIII-4-4}$$

proof:

We first prove theorem 4.2. We use again some Tutte like equations[1].

Consider the boundary containing x_1, and consider the first edge of that boundary, it has a sign $+$.

When we erase this edge, several situations may occur:

• on the other side of the removed edge, we have a j gon of sign $+$, then the corresponding term in Tutte equation will be:

$$ax_1H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)=\Big(\sum_j t_j x_1^{j-1}H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k)\Big)_{-}+\text{other possibilities}$$

[1]The proof of these two theorems was done in []. The proof presented here, is much simpler, and is due to Luigi Cantini in 2007,. It was never published and we thank L. Cantini for that proof.

where the subscript $()_-$ means that we keep only negative powers of x_1.

• on the other side of the removed edge, we have a bicolored $(+-)$ face, i.e. after removing the edge, we get an edge of sign $-$, which thus enters the boundary $y_{\pi'(1)}$, then the corresponding term in Tutte equation will be:

$$a x_1 H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = c\,(y_{\pi'(1)}\,H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k))_- + \text{other possibilities}$$

and the subscript $()_-$ means that we keep only negative powers of $y_{\pi'(1)}$.

• on the other side of the removed edge, we have an edge of the same face or of another marked face, let us say it is x_j for some $j\neq 1$. Erasing the edge either disconnects the boundary into two pieces (and reduces the genus by 1), or on the contrary merges two boundaries. In both cases, the boundary (π,π') becomes $(\pi,\pi'\,(1j))$. Then the corresponding term in Tutte equation will be:

$$\begin{aligned} a x_1 H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) &= \frac{t}{N}\,\frac{-1}{x_1-x_j}\Big(H_{\pi,\pi'(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k) \\ &\quad - H_{\pi,\pi'(1,j)}(x_j,x_2,\dots,x_k;y_1,\dots,y_k)\Big) \\ &\quad + \text{other possibilities} \end{aligned}$$

• on the other side of the removed edge, we have an edge of the x_1 component of the same boundary. Erasing the edge either disconnects the boundary into two pieces, which either disconnects the surface, or decreases the genus by 1. Then the corresponding term in Tutte equation will be:

$$\begin{aligned} a x_1 H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) &= W(x_1)\,H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) \\ &\quad + H^{(1)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;x_1) + \text{other possibilities} \end{aligned}$$

where $H^{(1)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;x_1)$ gathers all the possibilities of disconnecting the surface or decreasing the genus by 1.

Finally, writing that $ax - \sum_j t_j x^{j-1} = V_1'(x)$, that gives

$$\begin{aligned} &((V_1'(x_1) - W(x_1))H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k))_- \\ &- H^{(1)}_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k;x_1) \\ &- \frac{t}{N}\sum_{j\neq 1}\frac{1}{x_1-x_j}\,H_{\pi,\pi'(1,j)}(x_j,x_2,\dots,x_k;y_1,\dots,y_k) \\ =\;& c\,y_{\pi'(1)}\,H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) - \frac{t}{N}\sum_{j\neq 1}\frac{1}{x_1-x_j}\,H_{\pi,\pi'(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k) \\ =\;& c\sum_{\pi''} H_{\pi,\pi''}(x_1,\dots,x_k;y_1,\dots,y_k)_{\pi'',\pi'}\;(\mathcal{A}_1)_{\pi'',\pi'}\,. \end{aligned}$$

The key is to observe that all the terms in the left hand side are symmetric under transposition $\pi\leftrightarrow\pi'$, and thus, taking the difference of that equation with its transpose gives:

$$0 = H\mathcal{A}_1 - (H\,\mathcal{A}_1)^t = H\mathcal{A}_1 - \mathcal{A}_1 H = [H,\mathcal{A}_1].$$

The proof is similar for the other $\mathcal{A}_j$'s. This ends the proof of theorem 4.2.

We shall not prove theorem 4.3 here. We just give an argument towards it. The full proof is very involved and relies on group theory, so is beyond the scope of this book (more on the properties of matrices $\mathcal{M}$ can be found in [34, 75]).

The argument, is that the algebra of $k! \times k!$ matrices which commute with all $\mathcal{A}_i$'s is generated by the matrices $\mathcal{M}$.

Just observe that all those matrices commute together:

$$\forall \xi, \xi', \eta, \eta', \qquad [\mathcal{M}(x_1, \dots, x_k; y_1, \dots, y_k; \xi, \eta), \mathcal{M}(x_1, \dots, x_k; y_1, \dots, y_k; \xi', \eta')] = 0$$

(this commutation relation is not trivial, it relies on group theory of the unitary group $U(k)$ []).

Then, expanding $\mathcal{M}$ at large ξ and η, one has

$$\mathcal{A} = \frac{-Nc}{t} \operatorname*{Res}_{\xi\to\infty} \operatorname*{Res}_{\eta\to\infty} \xi\eta(\mathcal{M} - (1 + \frac{k(k-1)}{2} \frac{t^2}{N^2 c^2}) \,\mathrm{Id}_{k!})$$

which implies that $[\mathcal{A}, \mathcal{M}] = 0$.

Similarly, expanding at $\eta \to \infty$ and $\xi \to x_i$ we have

$$\mathcal{A}_i = \left(\frac{t}{Nc} \sum_{j\neq i} \frac{1}{x_i - x_j}\right) \mathrm{Id}_{k!} + \frac{Nc}{t} \operatorname*{Res}_{\xi\to x_i} \lim_{\eta\to\infty} \eta\, (\mathcal{M} - \mathrm{Id}),$$

which implies that $[\mathcal{A}_i, \mathcal{M}] = 0$.

There are also matrices $\mathcal{M}_{i,j}$ defined as

$$\mathcal{M}_{i,j}(x_1, \dots, x_k; y_1, \dots, y_k) = \operatorname*{Res}_{\xi\to x_i} \operatorname*{Res}_{\eta\to y_j} \mathcal{M}(x_1, \dots, x_k; y_1, \dots, y_k; \xi, \eta)$$

which also commute with all the others

$$[\mathcal{M}_{i,j}, \mathcal{M}] = 0.$$

□

Example $k = 2$

Let us see what this theorem tells us for $k = 2$:

We have

$$\mathcal{A} = \begin{pmatrix} x_1 y_1 + x_2 y_2 & \frac{-t}{Nc} \\ \frac{-t}{Nc} & x_1 y_2 + x_2 y_1 \end{pmatrix} = x_1 \mathcal{A}_1 + x_2 \mathcal{A}_2,$$

$$\mathcal{A}_1 = \begin{pmatrix} y_1 & \frac{-t}{Nc}\frac{1}{x_1 - x_2} \\ \frac{-t}{Nc}\frac{1}{x_1 - x_2} & y_2 \end{pmatrix} \quad , \quad \mathcal{A}_2 = \begin{pmatrix} y_2 & \frac{-t}{Nc}\frac{1}{x_2 - x_1} \\ \frac{-t}{Nc}\frac{1}{x_2 - x_1} & y_1 \end{pmatrix},$$

and we find that the matrix $\mathcal{M}$ is

$$\begin{aligned} \mathcal{M}(x_1, x_2; y_1, y_2; \xi, \eta) \;&=\; \left(1 - \frac{t}{Nc} \frac{2\xi\eta - \xi(y_1 + y_2) - \eta(x_1 + x_2) - \frac{t}{Nc}}{(x_1 - \xi)(y_1 - \eta)(x_2 - \xi)(y_2 - \eta)}\right) \mathrm{Id}_2 \\ &\quad - \frac{t}{Nc} \frac{1}{(x_1 - \xi)(y_1 - \eta)(x_2 - \xi)(y_2 - \eta)} \mathcal{A}, \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{M}_{1,1} &= \frac{-t}{N\,c}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{t^2}{N^2\,c^2}\,\frac{1}{(x_1-x_2)\,(y_1-y_2)}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\
&= \frac{t}{N\,c}\,\frac{x_1y_2+x_2y_1+\frac{t}{Nc}}{(x_1-x_2)(y_1-y_2)}\,\mathrm{Id}_2 - \frac{t}{N\,c}\,\frac{\mathcal{A}}{(x_1-x_2)(y_1-y_2)}.
\end{aligned}
$$

One easily verifies that they all commute together.

The eigenvalues of $\mathcal{A}$ are:

$$
\lambda = \frac{(x_1+x_2)(y_1+y_2)}{2} \pm \frac{1}{2}\sqrt{(x_1-x_2)^2\,(y_1-y_2)^2 + 4\,\frac{t^2}{N^2\,c^2}}.
$$

and the eigenvalues of $\mathcal{A}_1$ and $\mathcal{A}_2$ are

$$
\lambda_1 = \frac{y_1+y_2}{2} \pm \frac{1}{2(x_1-x_2)}\sqrt{(x_1-x_2)^2\,(y_1-y_2)^2 + 4\,\frac{t^2}{N^2\,c^2}}
$$

$$
\lambda_2 = \frac{y_1+y_2}{2} \mp \frac{1}{2(x_1-x_2)}\sqrt{(x_1-x_2)^2\,(y_1-y_2)^2 + 4\,\frac{t^2}{N^2\,c^2}}
$$

The common vectors of all these matrices, normalized so that $\sum_\pi (-1)^\pi v_\pi = 1$ are:

$$
v = \frac{1}{2\;x_{12}\,y_{12}}\begin{pmatrix}\frac{2t}{N\,c} + x_{12}\,y_{12} + \sqrt{x_{12}^2\,y_{12}^2 + 4\,\frac{t^2}{N^2\,c^2}}\\ \frac{2t}{N\,c} - x_{12}\,y_{12} + \sqrt{x_{12}^2\,y_{12}^2 + 4\,\frac{t^2}{N^2\,c^2}}\end{pmatrix}
$$

where we have denoted $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ to shorten the notations.

The matrix V with entries $V_{i,j} = \sum_{\pi,\,\pi(i)=j} (-1)^\pi\, v_\pi$, that we shall consider in the next section is:

$$
V = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{\frac{2t}{N\,c} + \sqrt{x_{12}^2\,y_{12}^2 + 4\,\frac{t^2}{N^2\,c^2}}}{2\;x_{12}\,y_{12}}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}
$$

Then write

$$
H(x_1,x_2;y_1,y_2) = \begin{pmatrix} H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1,x_2;y_1,y_2) & H_{\mathrm{Id}_2,(1.2)}(x_1,x_2;y_1,y_2)\\ H_{(1,2),\mathrm{Id}_2}(x_1,x_2;y_1,y_2) & H_{(1,2),(1.2)}(x_1,x_2;y_1,y_2)\end{pmatrix}
$$

That gives

$$
\begin{aligned}
&[\mathcal{M}(x_1,x_2;y_1,y_2;\xi,\eta), H(x_1,x_2;y_1,y_2)]\\
= &\Big((x_1-x_2)(y_2-y_1)H_{\mathrm{Id}_2,(1.2)}(x_1,x_2;y_1,y_2) + \frac{t}{N\,c}(H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1,x_2;y_1,y_2)\\
&-H_{(1,2),(1.2)}(x_1,x_2;y_1,y_2))\Big)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}
\end{aligned}
$$

and therefore theorem 4.3 implies

$$
H_{\mathrm{Id}_2,(1.2)}(x_1,x_2;y_1,y_2) = \frac{t}{N\,c}\,\frac{H_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1,x_2;y_1,y_2) - H_{(1,2),(1,2)}(x_1,x_2;y_1,y_2)}{(x_1-x_2)\,(y_1-y_2)}
$$

This can be rewritten in terms of connected $\widehat{H}$, and also written for fixed genus:

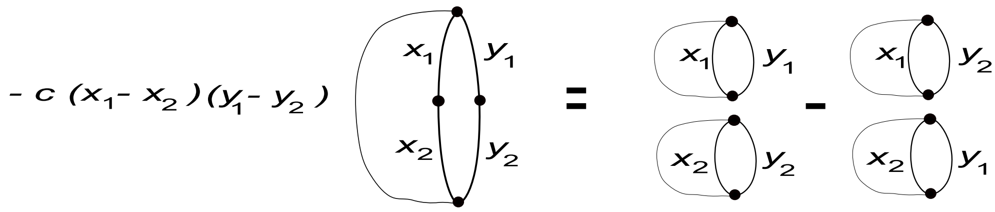
$$
\begin{aligned}
& c\,\widehat{H}^{(g)}_{\mathrm{Id}_2,(1.2)}(x_1,x_2;y_1,y_2) \\
= \;& \frac{\widehat{H}^{(g-1)}_{\mathrm{Id}_2,\mathrm{Id}_2}(x_1,x_2;y_1,y_2) - \widehat{H}^{(g-1)}_{(1,2),(1,2)}(x_1,x_2;y_1,y_2)}{(x_1-x_2)\,(y_1-y_2)} \\
& + \sum_{h=0}^{g} \frac{\widehat{H}^{(h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_1)\,\widehat{H}^{(g-h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_2) - \widehat{H}^{(h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_2)\,\widehat{H}^{(g-h)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_1)}{(x_1-x_2)\,(y_1-y_2)}.
\end{aligned}
$$

In particular, for the planar case $g=0$, that gives:

Corollary 4.1

$$
\begin{aligned}
& c\,\widehat{H}^{(0)}_{\mathrm{Id}_2,(1.2)}(x_1,x_2;y_1,y_2) \\
= \;& -\frac{\widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_1)\,\widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_2) - \widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_1;y_2)\,\widehat{H}^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_2;y_1)}{(x_1-x_2)\,(y_1-y_2)}.
\end{aligned}
\qquad (VIII-4-5)
$$

Graphically it says that:



At the time of writing of this book, there is no known combinatorical interpretation to that remarkable relationship.

Eigenvalues and eigenvectors of the commuting matrices $\mathcal{A}_i, \mathcal{A}, \mathcal{M}, \mathcal{M}_{i,j}$

The $k! \times k!$ matrices $\mathcal{A}_i$ all commute $[\mathcal{A}_i, \mathcal{A}_j] = 0$, and they also commute with $\mathcal{M}$, and thus they have a common basis of eigenvectors.

Finding the eigenvalues of a $k! \times k!$ matrix is not easy, and fortunately, the following theorem allows to find these eigenvalues, only in terms of $k \times k$ matrices, which is much easier:

Theorem 4.4 *Let $Y = \mathrm{diag}(y_1,\dots,y_k)$ and Ξ the $k\times k$ antisymmetric matrix $\Xi_{i,j} = \frac{t}{N}\frac{1}{x_i - x_j}$ if $i \neq j$, and $\Xi_{i,i} = 0$. Let $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_k)$ be a solution (there are $k!$ solutions to this algebraic equation) of*

$$\forall\, \eta\,, \qquad \det(\eta\,\mathrm{Id}_k - \Lambda - \Xi) = \det(\eta\,\mathrm{Id}_k - Y),$$

i.e. find Λ such that the eigenvalues of $\Lambda + \Xi$ are $y_1,\dots,y_k$

$$\mathrm{sp}(\Lambda + \Xi) = \{y_1,\dots,y_k\}.$$

Then, let $V_{i,j}$ be the $k\times k$ matrix of eigenvectors of $\Lambda+\Xi$

$$\Lambda+\Xi = V\,Y\,V^{-1}$$

normalized such that it is a stochastic matrix (it is proved below that this is indeed possible):

$$\sum_i V_{i,j} = \sum_j V_{i,j} = 1.$$

Then:

- *eigenvalue of* $\mathcal{A}_i$ *:* λ_i
- *eigenvalue of* $\mathcal{A}$ *:* $\lambda = \sum_i x_i\lambda_i$
- *eigenvalue of* $\mathcal{M}$ *:* $\mu = 1-\frac{t}{Nc}\,\sum_{i,j}\frac{V_{i,j}}{(\xi-x_i)(\eta-y_j)}$.
- *eigenvalue of* $\mathcal{M}_{i,j}$ *:* $\mu_{i,j} = -\frac{t}{Nc}\,V_{i,j}$

where $\mathcal{M}_{i,j} = \mathrm{Res}_{\,\xi\to x_i}\,\mathrm{Res}_{\,\eta\to y_j}\,\mathcal{M}$.

proof:

Let $v=(v_\pi)$ be a common eigenvector. Let us define the $k\times k$ matrix

$$V_{i,j} = \sum_{\pi\,|\,\pi(i)=j} (-1)^\pi\,\,v_\pi.$$

It satisfies:

$$\sum_i V_{i,j} = \sum_j V_{i,j} = \sum_\pi (-1)^\pi\,v_\pi = e^t.v$$

where e is the vector $e_\pi = (-1)^\pi$.

Notice that when N is large, or vice equivalently, when the x_i's and y_i's are large compared to t/Nc, the matrices $\mathcal{A}$ and $\mathcal{A}_i$'s are almost diagonal, with distinct eigenvalues. In this regime, the eigenvectors v tend to the basis vectors, i.e. only one component v_π is non-vanishing, i.e. $e^t.v\to(-1)^\pi v_\pi\neq 0$.

Moreover, the eigenvectors are algebraic functions of the x_i's and y_i's, and thus $e^t.v$ is an algebraic function, and it doesn't vanish in this regime, so it doesn't vanish for generic values of x_i's and y_i's.

We can thus chose to normalize our eigenvector v, for generic values of x_i's and y_i's, so that the matrix V is not identically vanishing, and so that:

$$e^t.v = 1.$$

The equation $\mathcal{A}_i\,v=\lambda_i v$ implies:

$$\forall\ i,j\,,\qquad \lambda_i V_{i,j} + \sum_{l\neq i}\frac{t}{Nc(x_i-x_l)}\,V_{l,j} = y_j\,V_{i,j}$$

If we define $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_k)$ and $Y=\mathrm{diag}(y_1,\dots,y_k)$, and $\Xi_{i,j}=\frac{t}{Nc(x_i-x_j)}$ and $\Xi_{i,i}=0$ we have

$$(\Lambda+\Xi)\,V = V\,Y.$$

If V would be invertible that would mean that the y_i's are the eigenvalues of $\Lambda + \Xi$:

$$\det(y - \Lambda - \Xi) = \prod_{i=1}^{k} (y - y_i),$$

or

$$\forall\, i\,, \qquad \det(y_i - \Lambda - \Xi) = 0.$$

In other words, we have the y_i's as functions of the λ_i's and x_i's. We can invert those relations and deduce the λ_i's as algebraic functions of the y_i's and x_i's.

Let $\tilde{Y} = (\tilde{y}_1, \dots, \tilde{y}_k)$ be the eigenvalues of $\Lambda + \Xi$, and let $\tilde{V}$ be a matrix whose columns are a basis of eigenvectors of $\Lambda+\Xi$. By definition the matrix $\tilde{V}$ of eigenvectors is invertible. The eigenvectors are defined up to a scalar factor, i.e. $\tilde{V}$ is defined up to right multiplication by an invertible diagonal matrix. For generic choices of x_i's and y_i's, we may normalize $\tilde{V}$ so that:

$$\forall\, j\,, \qquad \sum_i \tilde{V}_{i,j} = 1.$$

We thus have, by definition:

$$\Lambda + \Xi = \tilde{V}\tilde{Y}\tilde{V}^{-1}.$$

Multiplying by V on the right, and by $\tilde{V}^{-1}$ on the left, we get:

$$\tilde{Y}\tilde{V}^{-1}V = \tilde{V}^{-1}VY,$$

and Y and $\tilde{Y}$ are both diagonal matrices. Let $C = \tilde{V}^{-1}V$, we have:

$$\forall\, i,j\,, \qquad C_{i,j}\,(\tilde{y}_i - y_j) = 0.$$

This implies that either $C_{i,j} = 0$, or $\tilde{y}_i = y_j$. Moreover we have, by our choices of normalization, that

$$\forall\, j\,, \qquad \sum_i C_{i,j} = 1$$

so that for each j there must exist some i with $C_{i,j} \neq 0$, and thus there must exist some i with $\tilde{y}_i = y_j$. If the y_j's are all distinct, then there is at most one $\tilde{y}_i$ equal to y_j for each j, and thus $C_{i,j} = 0$ for all the others. Up to reordering the eigenvalues of $\tilde{Y}$, we may chose that $\tilde{y}_i = y_i$, and C must be diagonal, and since $\sum_i C_{i,j} = 1$, we must have $C = \mathrm{Id}$:

$$\tilde{Y} = Y \qquad , \qquad V = \tilde{V}.$$

This proves in particular that V is invertible (for generic values of x_i's and y_i's).

So we have proved that V is the matrix of eigenvectors of $\Lambda+\Xi$, and can be chosen to be stochastic.

Then, if we write

$$\mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta) = \mathrm{Id} + \sum_{i,j} \frac{1}{(\xi - x_i)\,(\eta - y_j)}\,\mathcal{M}_{i,j}(x_1,\dots,x_k;y_1,\dots,y_k)$$

where

$$\mathcal{M}_{i,j}(x_1,\dots,x_k;y_1,\dots,y_k) = \mathop{\mathrm{Res}}_{\xi\to x_i}\ \mathop{\mathrm{Res}}_{\eta\to y_j} \mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta)$$

Let us call μ the eigenvalue of $\mathcal{M}$ and $\mu_{i,j}$ the eigenvalues of $\mathcal{M}_{i,j}$ for the eigenvector v.

$$\mathcal{M}(x_1,\dots,x_k;y_1,\dots,y_k;\xi,\eta)\,v = \mu\, v \qquad , \qquad \mathcal{M}_{i,j}(x_1,\dots,x_k;y_1,\dots,y_k)\,v = \mu_{i,j}\, v$$

we have:

$$\mu = 1 + \sum_{i,j} \frac{\mu_{i,j}}{(\xi-x_i)(\eta-y_j)}.$$

Let us multiply on the left by the vector e of components $e_\pi = (-1)^\pi$, we get

$$e^t\, \mathcal{M}\, v = \mu\, e^t v.$$

Let us compute $e^t\mathcal{M}$:

$$\begin{aligned}
(e^t\,\mathcal{M})_\pi &= \sum_{\pi'} (-1)^{\pi'} \prod_i \left(\delta_{\pi'(i),\pi(i)} - \frac{t}{Nc}\,\frac{1}{(\xi-x_i)(\eta-y_{\pi'(i)})} \right) \\
&= \det\left(\delta_{j,\pi(i)} - \frac{t}{Nc}\,\frac{1}{(\xi-x_i)(\eta-y_j)} \right) \\
&= (-1)^\pi\ \det\left(\delta_{i,j} - \frac{t}{Nc}\,\frac{1}{(\xi-x_i)(\eta-y_{\pi(j)})} \right)
\end{aligned}$$

Notice that the matrix inside the determinant is of the form $\mathrm{Id} - AB^t$, where A and B are vectors, we have:

$$\det(\mathrm{I} - AB^t) = 1 - B^t A,$$

and thus

$$(e^t\,\mathcal{M})_\pi = (-1)^\pi\ \left(1 - \frac{t}{Nc} \sum_i \frac{1}{(\xi-x_i)(\eta-y_{\pi(i)})} \right),$$

and then, taking the residue at $\xi = x_i$ and $\eta = y_j$ gives

$$(e^t\,\mathcal{M}_{i,j})_\pi = -\frac{t}{Nc}\ (-1)^\pi\ \delta_{\pi(i),j}.$$

Multiplying $\mathcal{M}_{i,j} v = \mu_{i,j} v$ by e^t on the left thus gives

$$-\frac{t}{Nc}\ V_{i,j} = \mu_{i,j}\ e^t.v = \mu_{i,j}\ .$$

In that case, we have that the eigenvalue of $\mathcal{M}_{i,j}$ is $\mu_{i,j} = -\frac{t}{Nc}\, V_{i,j}$.

$$\mathcal{M}_{i,j}\, v = -\frac{t}{Nc}\ V_{i,j}\, v.$$

This ends the proof of the theorem.

$\square$

Corollary 4.2 *Since the matrices $H(x_1,\dots,x_k;y_1,\dots,y_k)$ commute with $\mathcal{M}$'s, they must have the same basis of eigenvectors. Let $\mathcal{V}_{\pi,\rho}$ be a matrix whose columns are eigenvectors normalized so that:*

$$\mathcal{V}^t\mathcal{V} = \mathrm{Id}$$

(which is possible since all our matrices are symmetric), and so that

$$\mathcal{V}e = e \qquad , \qquad e^t\mathcal{V} = e^t$$

where e is the vector $e_\pi = (-1)^\pi$.

We thus may write:

$$H_{\pi,\pi'}(x_1,\dots,x_k;y_1,\dots,y_k) = \sum_\rho \mathcal{V}_{\pi,\rho}\,\mathcal{V}_{\pi',\rho}\;\mathcal{H}_\rho(x_1,\dots,x_k;y_1,\dots,y_k)$$

where the $\mathcal{H}_\rho(x_1,\dots,x_k;y_1,\dots,y_k)$'s are the eigenvalues of $H(x_1,\dots,x_k;y_1,\dots,y_k)$, indexed by a permutation ρ.

This can also be written as:

$$\mathcal{H} = \mathcal{V}^t\,H\,\mathcal{V} \qquad \text{is diagonal.}$$

4.2 Planar discs

Here we restrict ourselves to the $g = 0$ planar case, and also to the case where we have only one boundary, i.e. $\pi'^{-1}\circ\pi$ has only one cycle, and up to renaming the variables, we can always choose $\pi = \mathrm{Id}_k$ and π' as the shift $\pi'(i) = i - 1 \bmod k$, i.e. $\pi' = (1 \to k \to k-1 \to k-2 \to \dots \to 2 \to 1)$. Our goal in this section is to compute explicitly all the generating functions

$$H^{(0)}_{\mathrm{Id}_k,(1\to2\to\dots\to k\to1)}(x_1,\dots,x_k;y_1,\dots,y_k).$$

This is achieved by the following theorem, and it uses the knowledge of the spectral curve $E(x,y) = 0$, which we have written parametrically in section 3 above as $x = x(z), y = y(z)$, or in theorem 3.2:

Theorem 4.5 *for $k = 1$*

$$H^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x(z);y(z')) = \frac{-1}{c}\,\frac{E(x(z),y(z'))}{(x(z)-x(z'))\,(y(z)-y(z'))}, \tag{VIII-4-6}$$

and for $k > 1$:

$$\begin{aligned} &H^{(0)}_{\mathrm{Id}_k,(1\to2\to\dots\to k\to1)}(x_1,\dots,x_k;y_1,\dots,y_k) \\ &= \sum_\sigma C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k)\,\prod_{i=1}^k H^{(0)}_{\mathrm{Id}_1,\mathrm{Id}_1}(x_i;y_{\sigma(i)}) \end{aligned} \tag{VIII-4-7}$$

where the coefficients $C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k)$ are some universal rational functions (defined further below) of the x_i's and y_i's, they are independent of the parameters t_k's and $c_{\pm,\pm}$.

Again, the proof of this theorem is very technical and far from straightforward, and we refer the motivated reader to []. Eq.(VIII-4-7) can also be derived from theorem 4.3.

proof:

Let us first prove eq.(VIII-4-6) by using Tutte's method again.

Recall that we have computed the generating function (see def.2.2)

$$U_0^{(0)}(x,y) = -V_2'(y) + cx - \sum_{j=2}^{\tilde{d}} \sum_{k=0}^{j-2-k} G_{0,k}^{(0)}(x)$$

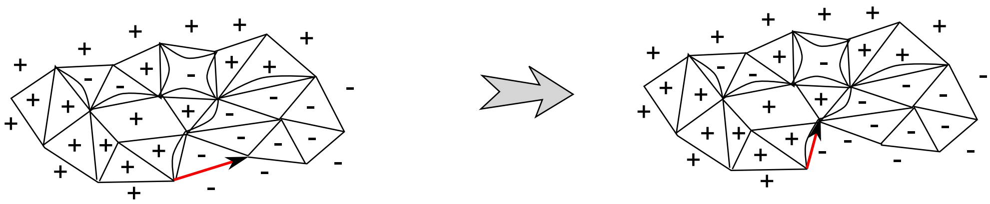
which counts planar maps with 1 mixed boundary, made of a $-$ boundary of length k and a $+$ boundary of arbitrary length weighted by $x^{-1-\mathrm{length}}$, and in eq.(VIII-3-1) we have found:

$$U_0^{(0)}(x,y) = \frac{E(x,y)}{c(y-Y(x))}.$$

Then, consider a planar Ising map with a unique marked face with $(+,-)$ boundary with arbitrary lengths.

Chose the first $-$ edge on the boundary, and we shall erase it. Several possibilities may occur:

- on the other side of the removed edge, we have a j gon of sign $-$



then the corresponding term in Tutte equation will be:

$$by(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) = \sum_j \tilde{t}_j y^{j-1}(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)) + \text{other possibilities}$$

Notice that erasing the edge can be done only if the length is positive, i.e. if the power of y is strictly negative, which we can write

$$b\left(yH_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)\right)_- = \left(\sum_j \tilde{t}_j y^{j-1} H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)\right)_- + \text{other possibilities}$$

Recall that by definition of V_2'

$$by - \sum_j \tilde{t}_j y^{j-1} = V_2'(y),$$

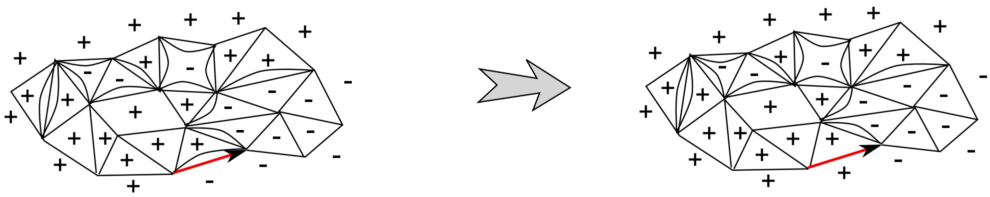
and observe that the positive powers of y in $V_2'(y)H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y)$ is precisely the definition eq.(VIII-2-3) in section 2.1 of $U_0^{(0)}(x,y) + cV_2'(y) - c^2x$:

$$\left(V_2'(y)(c + H_{\mathbf{1},\mathbf{1}}^{(0)}(x;y))\right)_+ = U_0^{(0)}(x,y) + cV_2'(y) - c^2x,$$

and thus the equation is

$$V_2'(y)(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)) = U_0^{(0)}(x,y) + cV_2'(y) - c^2 x + \text{other possibilities}$$

• on the other side of the removed edge, we have a bicolored $(+-)$ face, i.e. after removing the edge, we get an edge of sign $+$,



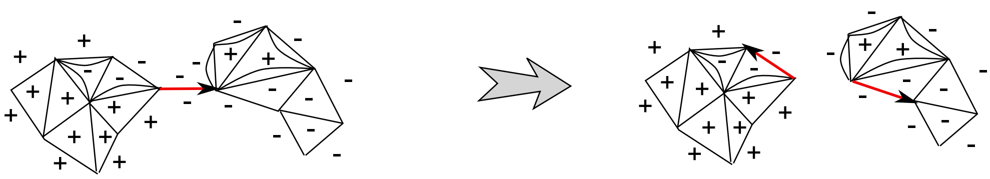
the corresponding term in Tutte equation will be:

$$b\left(y(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y))\right)_- = c\left(x\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y))\right)_- + \text{other possibilities}$$

where in the right hand side we need to keep only strictly negative powers of x. Notice that the positive powers of x in $x\,H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)$ give $\tilde{W}(y) = V_2'(y) - cX(y)$, and thus we get

$$\left(by(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y))\right)_- = cx\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)) - c(V_2'(y)-cX(y)) + \text{other possibilities}$$

• on the other side of the removed edge, we have the same face



then the corresponding term in Tutte equation will be:

$$\begin{aligned}\left(by(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y))\right)_- &= \tilde{W}(y)\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)) + \text{other possibilities} \\ &= (V_2'(y)-cX(y))\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)) + \text{other possibilities}\end{aligned}$$

Finally, putting all possibilities together we get

$$\begin{aligned}V_2'(y)\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)) &= U_0^{(0)}(x,y) + cV_2'(y) - c^2 x \\ &\quad + cx\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y)) - c(V_2'(y)-cX(y)) \\ &\quad + (V_2'(y)-cX(y))\,(c+H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y))\end{aligned}$$

many terms cancel and it remains

$$c(X(y)-x))\,H^{(0)}_{\mathbf{1},\mathbf{1}}(x;y) = U_0^{(0)} = \frac{E(x,y)}{y-Y(x)}$$

This proves the first part of the theorem.

Then we use:

$$[H,\mathcal{A}_1]_{\pi,\pi'} = 0$$

with $\pi = \mathrm{Id}_k$ and $\pi' = S_k$. This gives:

$$\begin{aligned}
& \frac{Nc}{t}\,(y_1-y_k)\,H_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k) \\
= & \sum_{j\neq 1}\frac{1}{x_1-x_j}\left(H_{(1,j),S_k}(x_1,\dots,x_k;y_1,\dots,y_k) - H_{\mathrm{Id}_k,S_k(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k)\right)
\end{aligned}$$

and keep only planar terms:

$$H_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k) \to \frac{N}{t}\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k)$$

and

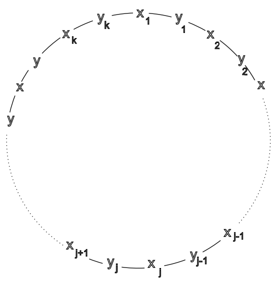
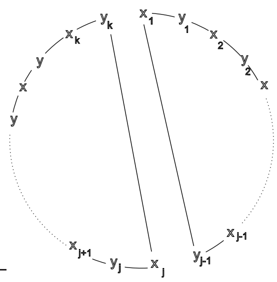
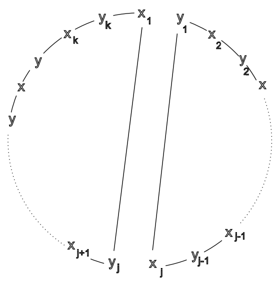
$$\begin{aligned}
H_{(1,j),S_k}(x_1,\dots,x_k;y_1,\dots,y_k) \;\to\; & \frac{N^2}{t^2}\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_j,x_2,\dots,x_{j-1};y_1,\dots,y_{j-1}) \\
& \hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_1,x_{j+1},\dots,x_k;y_j,\dots,y_k)
\end{aligned}$$

and

$$\begin{aligned}
H_{\mathrm{Id}_k,S_k\circ(1,j)}(x_1,\dots,x_k;y_1,\dots,y_k) \;\to\; & \frac{N^2}{t^2}\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_2,\dots,x_j;y_2,\dots,y_j) \\
& \hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_1,x_{j+1},\dots,x_k;y_1,y_{j+1},\dots,y_k)
\end{aligned}$$

$$\begin{aligned}
& c\,(y_k-y_1)\,\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k) \\
= & \sum_{j\neq 1}\frac{1}{x_1-x_j}\Big(\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_1,\dots,x_{j-1};y_1,\dots,y_{j-1}) \quad \hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_j,\dots,x_k;y_j,\dots,y_k) \\
& -\hat{H}^{(0)}_{\mathrm{Id}_{j-1},S_{j-1}}(x_j,x_2,\dots,x_{j-1};y_1,\dots,y_{j-1}) \quad \hat{H}^{(0)}_{\mathrm{Id}_{k+1-j},S_{k+1-j}}(x_1,x_{j+1},\dots,x_k;y_j,\dots,y_k)\Big)
\end{aligned}$$

This can be illustrated as:

$c(y_k - y_1)$  $= \sum_j \frac{1}{x_1 - x_j}$  $-$ 

We see that at each step we split the set of variables $\{x_i\}$'s and $\{y_i\}$'s into disjoint subsets, by drawing two arcs, which split the circle into two circles. The 2 arcs can never cross.

By an easy recursion, we shall eventually split the circle by a set of arcs, in order to reach only circles of length 2, i.e. a product of $\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_i; y_{\sigma(i)})$ with σ a permutation. Moreover, σ must be a "planar" permutation, i.e. it draws a link pattern on the circle, which can never cross itself.

Therefore there exists some coefficients C_σ's which are rational functions of the x_i's and of the y_i's, such that

$$\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k) = \sum_{\sigma\in\mathfrak{S}_k} C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k)\,\prod_{i=1}^{k}\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_i;y_{\sigma(i)}).$$

where $C_\sigma = 0$ if σ is not planar.

The coefficients C_σ satisfy the recursion:

$$C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k) = \frac{1}{c\,(y_k-y_1)}\sum_{j=2}^{k} C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k)\,C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k)$$

This ends the proof. The coefficients C_σ are computed below. $\square$

• Example $k=2$:

$$\hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_1,x_2;y_1,y_2) = \frac{\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\quad\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_2) - \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_2)\quad\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)}{c\,y_{21}x_{12}}$$

• Example $k=3$:

$$\begin{aligned}
&\hat{H}^{(0)}_{\mathrm{Id}_3,S_3}(x_1,x_2,x_3;y_1,y_2,y_3)\\
=\;&\frac{\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\;\hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_2,x_3;y_2,y_3) - \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)\;\hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_1,x_3;y_2,y_3)}{c\,x_{12}\,y_{31}}\\
&+\frac{\hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_1,x_2;y_1,y_2)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_3) - \hat{H}^{(0)}_{\mathrm{Id}_2,S_2}(x_3,x_2;y_1,y_2)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_3)}{c\,x_{13}\,y_{31}}
\end{aligned}$$

which gives:

$$\begin{aligned}
&c^2\,\hat{H}^{(0)}_{\mathrm{Id}_3,S_3}(x_1,x_2,x_3;y_1,y_2,y_3)\\
=\;&\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_2)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_3)\;\frac{1}{x_{12}y_{31}}\left(\frac{1}{x_{23}y_{32}}+\frac{1}{x_{13}y_{21}}\right)\\
&+\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_3)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_2)\;\frac{1}{x_{13}y_{31}}\left(\frac{1}{x_{12}y_{32}}+\frac{1}{x_{32}y_{21}}\right)\\
&+\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_1)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_3)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_2)\;\frac{1}{x_{12}x_{23}y_{23}y_{31}}\\
&+\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_3)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_2)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_1)\;\frac{1}{x_{13}x_{32}y_{21}y_{13}}\\
&+\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_1;y_2)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_2;y_1)\;\hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_3;y_3)\;\frac{1}{x_{21}x_{13}y_{32}y_{21}}
\end{aligned}$$

The planar link patterns and the coefficients C_σ

In the planar case, the boundary generating functions are thus of the form:

$$\hat{H}^{(0)}_{\mathrm{Id}_k,S_k}(x_1,\dots,x_k;y_1,\dots,y_k)=\sum_{\sigma\in\mathfrak{S}_k} C_\sigma(x_1,\dots,x_k;y_1,\dots,y_k)\,\prod_{i=1}^k \hat{H}^{(0)}_{\mathbf{1},\mathbf{1}}(x_i;y_{\sigma(i)}).$$

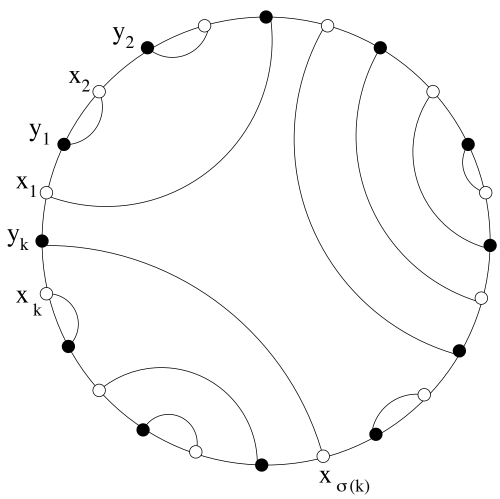
The coefficients C_σ satisfy a recursion relation. The explicit solution of this recursion was found in [], and we just mention the result.

The coefficients C_σ are determined as follows (we recall that $\pi=\mathrm{Id}_k$ and $\pi'=S_k$ is the shift $\pi'(i)=i-1 \bmod k$, and $\ell(\sigma)$ denotes the number of cycles of a permutation σ):

• C_σ vanishes if $\ell(\sigma^{-1}\circ\pi)+\ell(\sigma^{-1}\circ\pi')-\ell(\pi'^{-1}\circ\pi)\neq k$:

$$C_\sigma\neq 0 \quad\Rightarrow\quad \ell(\sigma^{-1}\circ\pi)+\ell(\sigma^{-1}\circ\pi')-\ell(\pi'^{-1}\circ\pi)=k \tag{VIII-4-8}$$

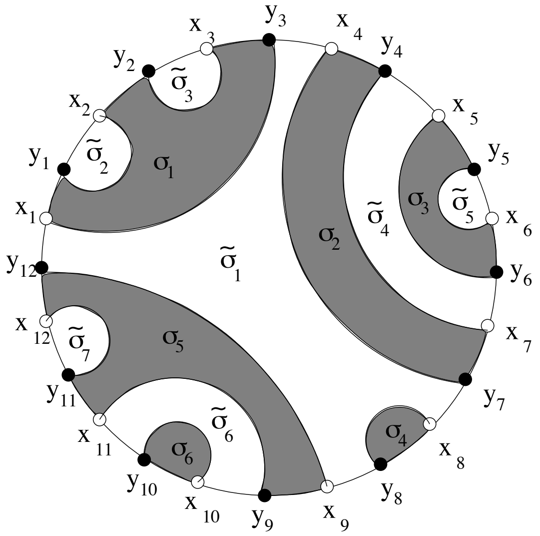
This means that σ must be a planar link pattern drawn on the cycles of $\pi'^{-1}\circ\pi$:



• If condition eq.(VIII-4-8) is satisfied, we decompose the cycles of $\sigma^{-1}\circ\pi$ and $\sigma^{-1}\circ\pi'$ as follows:

$$\sigma^{-1}\circ\pi=\prod_{i=1}^m \sigma_i \qquad , \qquad \sigma^{-1}\circ\pi'=\prod_{i=1}^{m'} \bar\sigma_i$$

In other words, each σ_i or $\bar{\sigma}_i$ is a face of the link pattern.



If j is in a cycle σ_i of $\sigma^{-1} \circ \pi$, the face is

$$x_j \to y_{\pi(j)} \to x_{\sigma_i(j)} \to y_{\pi(\sigma_i(j))} \to x_{\sigma_i(\sigma_i(j))} \to \ldots \to y_{\sigma^{-1}(j)} \to x_j$$

and we prefer to write σ_i as the ordered set of variables:

$$\sigma_i \equiv \left(x_j, y_{\pi(j)}, x_{\sigma_i(j)}, y_{\pi(\sigma_i(j))}, x_{\sigma_i(\sigma_i(j))}, \ldots, y_{\sigma^{-1}(j)}\right)$$

Similarly for $\bar{\sigma}_i$, we write

$$\bar{\sigma}_i \equiv \left(x_j, y_{\pi'(j)}, x_{\bar{\sigma}_i(j)}, y_{\pi'(\bar{\sigma}_i(j))}, x_{\bar{\sigma}_i(\bar{\sigma}_i(j))}, \ldots, y_{\sigma^{-1}(j)}\right)$$

With those notations, we shall write C_σ as a product of faces:

$$C_\sigma = \prod_{i=1}^{m} F_{\ell(\sigma_i)}(\sigma_i) \ \prod_{i=1}^{m'} F_{\ell(\bar{\sigma}_i)}(\bar{\sigma}_i).$$

The "face" functions $F_k(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$ are defined by the following recursion:

$$F_1(x, y) = 1$$

and

$$F_k(x_1, y_1, \ldots, x_k, y_k) = \sum_{j=1}^{k-1} \frac{F_j(x_1, y_1, \ldots, x_j, y_j)\, F_{k-j}(x_{j+1}, y_{j+1}, \ldots, x_k, y_k)}{c\,(x_1 - x_k)\,(y_k - y_j)}$$

They have the property to be cyclically invariant.

For instance for $k = 2$ we get

$$F_2(x_1, y_1, x_2, y_2) = \frac{1}{c\,(x_1 - x_2)\,(y_2 - y_1)}.$$

In fact, it is possible to write the functions F_k's as sums over trees, this was done in [], and we refer the interested reader to that article.

5 Summary: Ising model

Let us summarize the concepts introduced in this chapter:

- The Ising model is the combinatorics of bi-colored maps (colors = + and −, also called spin), conditionned on the number of edges separating faces of same or different colors.

  We attach Boltzman weights t_k for the number of $k-$gons of color $+$, $\tilde{t}_k$ for the number of $k-$gons of color $-$, and c_{++} for $++$ edges, c_{--} for $--$ edges, and c_{+-} for $+-$ edges. We define $c = c_{+-}/(c_{++}c_{--} - c_{+-}^2)$, $a = c_{--}/(c_{++}c_{--} - c_{+-}^2)$, $b = c_{++}/(c_{++}c_{--} - c_{+-}^2)$.

  These define the potentials:

$$V_1(x) = ax - \sum_{k\geq 3} \frac{t_k}{k} x^k$$

$$V_2(y) = by - \sum_{k\geq 3} \frac{\tilde{t}_k}{k} y^k$$

- One can write Tutte equations by recursively erasing the marked edge of the first marked face.

- The disc amplitude $W_1^{(0)}(x)$ is an algebraic function, we define $Y(x) = \frac{1}{c}(V_1'(x) - W_1^{(0)}(x))$. It satisfies an algebraic equation:

$$E(x, Y) = 0$$

  where $E(x, y)$ is a polynomial of its 2 variables, of degree $d_1 = \deg V_1$ in x and $d_2 = \deg V_2$ in y.

$$E(x, y) = (V_1'(x) - cy)(V_2'(y) - cx) - \frac{1}{c} P_0^{(0)}(x, y) + tc$$

  where $P_0^{(0)}(x, y)$ is a polynomial of degree at most $\deg V_1''$ in x and $\deg V_2''$ in y.

  The polynomial $P_0^{(0)}(x, y)$ is uniquely determined by requiring that the algebraic equation $E(x, y) = 0$ defines a Riemann surface of genus 0, and by $P_0^{(0)}(x, y) = \frac{V_1'(x)}{x}\frac{V_2'(y)}{y} + O(t)$

  Since the algebraic equation has genus zero, one can find a parametric solution with rational functions:

$$\begin{cases} x = x(z) = \gamma z + \sum_{k=0}^{\deg V_2'} \alpha_k z^{-k} \\ Y = y(z) = \gamma z^{-1} + \sum_{k=0}^{\deg V_1'} \beta_k z^k \end{cases}$$

and the coefficients γ, α_k, β_k's are uniquely determined by requiring that

$$V_1'(x(z)) - cy(z) \underset{z\to\infty}{\sim} \frac{t}{\gamma z} + O(z^{-2})$$

$$V_2'(y(z)) - cx(z) \underset{z\to 0}{\sim} \frac{t\,z}{\gamma} + O(z^2),$$

and by $\gamma^2 = O(t)$.

The disc amplitude is then:

$$W_1^{(0)}(x(z)) = V_1'(x(z)) - cy(z).$$

- The cylinder amplitude $W_2^{(0)}$.

  In the z variables (i.e. writing $x_i = x(z_i)$), we have:

$$W_2^{(0)}(x_1, x_2) = \frac{1}{(z_1-z_2)^2\; x'(z_1)\, x'(z_2)} - \frac{1}{(x_1-x_2)^2}.$$

  The differential form:

$$B(z_1, z_2) = W_2^{(0)}(x_1, x_2) dx_1 d_2 + \frac{dx_1 dx_2}{(x_1-x_2)^2} = \frac{dz_1 dz_2}{(z_1-z_2)^2}$$

  is called the fundamental 2nd kind differential.

  The cylinder amplitude is thus universal, in the z variables it is always the fundamental 2nd kind differential.

- Higher topology amplitudes are given by the topological recursion.

- We also find algebraic formulae for enumerating maps with multi-colored boundaries.

6 Exercises

Exercise 1: For Ising quadrangulations (only t_4 and $\tilde{t}_4$ non vanishing), count elements of $\mathbb{M}_0^{(0)}(2)$ and $\mathbb{M}_0^{(0)}(3)$, i.e. Ising quadrangulations with 1 boundary (of color +), and with 2 and 3 vertices.

Answer:

$$W_1^{(0)} = \frac{t}{x} + t^3 \left(\frac{2c_{++}^2}{x^5} + \frac{2t_4 c_{++}^3 + 2\tilde{t}_4 c_{+-}^2 c_{--}}{x^3} \right) + O(t^4)$$

Exercise 2: For Ising quadrangulations (only t_4 and $\tilde{t}_4$ non vanishing), find the disc amplitude. Write the parametrization:

$$x(z) = \gamma z + \alpha_1 z^{-1} + \alpha_3 z^{-3},$$

$$y(z) = \gamma z^{-1} + \beta_1 z + \beta_3 z^3.$$

Find that the algebraic equation satisfied by γ^2 is of degree 5:

$$t = \frac{a\gamma^2}{c} \frac{b - 3a\tilde{t}_4 \frac{\gamma^2}{c}}{1 - 9t_4\tilde{t}_4 \frac{\gamma^4}{c^2}} - c\gamma^2 \left(1 - 3t_4\tilde{t}_4 \frac{\gamma^4}{c^2}\right) - 3t_4 \frac{\gamma^4}{c^2} \left(\frac{b - 3a\tilde{t}_4 \frac{\gamma^2}{c}}{1 - 9t_4\tilde{t}_4 \frac{\gamma^4}{c^2}}\right)^2$$

Consider the special case where $t_4 = \tilde{t}_4$, $a = b$ and $c = 1$. In that case we shall have $\alpha_1 = \beta_1$ and $\alpha_3 = \beta_3$. Find the equation for γ^2:

$$t = \frac{a^2\gamma^2}{1 + 3t_4\gamma^2} - \gamma^2 \left(1 - 3t_4^2\gamma^4\right) - \frac{3a^2 t_4 \gamma^4}{(1 + 3t_4\gamma^2)^2}$$