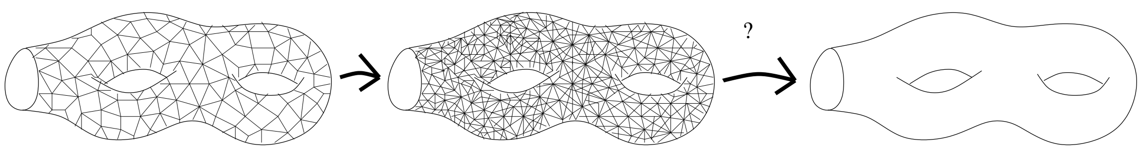
Chapter V

Counting large maps

Initially, in quantum gravity and string theory, the problem of counting maps, i.e. surfaces made of polygons, was introduced only as a discretized approximation for counting continuous surfaces. The physical motivation is the following: in string theory, particles are 1-dimensional loops called strings, and under time evolution their trajectories in space-time are surfaces. Quantum mechanics amounts to averaging over all possible trajectories between given initial and final states, i.e. all possible surfaces between given boundaries. However, trajectories should be counted only once modulo their symmetries, in particular conformal reparametrizations, in other words, trajectories are in fact Riemann surfaces (equivalence class of surfaces modulo conformal reparametrizations).

The set of all Riemann surfaces with a given topology and given boundaries, is called the moduli space, and string theory amounts to "counting" Riemann surfaces, i.e. measuring the "volume" of the moduli space.

Physicists made the guess that in some appropriate limit, the counting function of discrete surfaces (maps) should tend towards the counting function of Riemann surfaces. In some sense, **surfaces made of a very large number of very small polygons should be a good approximation of Riemann surfaces in quantum gravity** !



In this chapter, we are going to explain how to find the asymptotic generating functions of large maps, and then compare with Liouville conformal field theory of quantum gravity, and in the next chapter we are going to compare it to the enumeration of Riemann surfaces.

1 Introduction to large maps and Double scaling limit

The idea is to count maps made of a very large number of polygons, and send the size of polygons (the mesh) to zero so that the average area remains finite.

1.1 Large size asymptotics and singularities

Let us start with general considerations about large order behaviors.

It is a standard knowledge that there is a relationship between the large order behavior of a sequence, and singularities of the corresponding generating series. Consider a sequence $\{A_k\}_{k\in\mathbb{N}}$, and the formal series:

$$A(t) = \sum_{k=0}^{\infty} A_k\, t^k.$$

Imagine that $A(t)$ is convergent in a disc $|t| < |t_c|$, for instance assume that it is an algebraic function of t (which is indeed the case for generating functions for maps).

The basic example is:

$$A(t) = C\ (t_c - t)^{-\alpha} = C\ t_c^{-\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)}\,(t/t_c)^k$$

The large order behavior is obtained from Stirling's asymptotic formula:

$$A_k = C\ t_c^{-\alpha-k}\,\frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} \mathop{\sim}_{k\to\infty} C\ \frac{t_c^{-\alpha}}{\Gamma(\alpha)}\ t_c^{-k}\ k^{\alpha-1} \tag{V-1-1}$$

More generally, if $A(t)$ is an analytical function with several algebraic singularities $t_{c1}, t_{c2}, t_{c3}, \ldots$ with exponents $\alpha_1, \alpha_2, \alpha_3, \ldots$, the large order behavior of A_k is dominated by the singularity(ies) t_{ci} closest to the origin, those for which $|t_{ci}|$ is minimal.

$$A_k \mathop{\sim}_{k\to\infty} \sum_{|t_{ci}|=\min\{|t_{cj}|\}} C_i\,\frac{t_{ci}^{-\alpha_i}}{\Gamma(\alpha_i)}\ t_{ci}^{-k}\ k^{\alpha_i-1}$$

Conversely, if a sequence A_k has a large order behavior of type eq.(V-1-1) with α rational, then its generating series $A(t)$ has a singularity of algebraic type.

There is also an intuitive approach to understand the link between singularities and large order behaviors. The expectation value of k is:

$$< k >= \frac{\sum_k k\ \ A_k\, t^k}{\sum_k A_k\, t^k} = \frac{t\,A'(t)}{A(t)}$$

thus, if we want large values of k to dominate the expectation values, i.e. if we want $< k >$ to become very large, we need to choose t such that tA'/A diverges, that is we need to choose t close to a point where $\ln A(t)$ is not analytical.

A weaker statement would be to require that some moment of k diverges, for instance:

$$< k^p >= \frac{1}{A(t)}\,\sum_k k^p\;\; A_k\, t^k = \frac{1}{A(t)}\,\left(t\,\frac{d}{dt}\right)^p\; A(t)$$

In other words we want to choose $t = t_c$ such that some derivative of $A(t)$ diverges.

Let us now illustrate those general considerations on some examples.

1.2 Example: quadrangulations

The generating function of quadrangulations of genus g with n_4 quadrangular unmarked faces, and thus $v = n_4 + 2 - 2g$ vertices is:

$$F_g(t_4) = t^{2-2g}\,\sum_{n_4} (t\,t_4)^{n_4} \sum_{\Sigma \in \mathbb{M}_0^{(g)}(n_4+2-2g)} \frac{1}{\#\mathrm{Aut}(\Sigma)}.$$

The average number of faces is thus:

$$< n_4 >= t_4 \frac{\partial \ln F_g}{\partial t_4} =< v > +2g - 2 = t\frac{\partial \ln F_g}{\partial t} + 2g - 2$$

where $< v >$ is the average number of vertices.

In order to have $< n_4 >$ or $< v >$ very large, one must chose t in the vicinity of a singularity of F_g. We have seen in chapter III, that all the F_g's (except F_0 and F_1) are rational fractions of $\gamma^2 = \frac{1-\sqrt{1-12tt_4}}{6t_4}$, and thus F_g is singular when γ^2 is singular, that is at $t = t_c = 1/12\,t_4$. For instance, with the notation $r = \sqrt{1-12tt_4}$, we have according to eq.(III-6-1), eq.(III-6-2) and eq.(III-6-3)):

$$\begin{aligned} F_0 &= \frac{t^2}{2}\left(\frac{1}{3(1+r)^2} - \frac{5}{3(1+r)} + \frac{3}{4} - \ln\frac{1+r}{2}\right) \\ &= \frac{t^2}{2}\left(\ln 2 - \frac{1}{3} - 4tt_4 + 36t^2t_4^2\right) - \frac{4}{15}\,t^2\,(1-t/t_c)^{5/2} + O((1-t/t_c)^3), \end{aligned}$$

$$F_1 = \frac{1}{12}\ln\frac{1+r}{2r} = -\frac{1}{24}\ln(1-12tt_4) - \frac{\ln 2}{12} + O((1-t/t_c)^{1/2}),$$

$$\begin{aligned} F_2 &= t^{-2}\left(\frac{-89r^5 + 20r^4 + 130r^3 - 100r^2 - 65r + 56}{5*9*2^8\,r^5} - \frac{B_4}{8}\right) \\ &= \frac{7}{10\;\;(12\,t_c)^2}\,(1-t/t_c)^{-5/2} + O(1-t/t_c)^{-2} \end{aligned}$$

Below, we will prove in theorem 3.1 that in general, for quadrangulations, F_g is singular at $t = t_c = 1/12\,t_4$, and behaves (for $g \geq 2$) like:

$$F_g \sim \tilde{F}_g\; t_c^{2-2g}\,(1-t/t_c)^{\frac{5}{4}(2-2g)} + \ldots \text{subleading}$$

the constant prefactor $\tilde{F}_g$ is called the "**double scaling limit**" of F_g, and our main goal from now on, is to compute it, not only for quandrangulations, but for all sorts of maps. We address that problem below, and the answer is given in theorem 3.1.

For F_1 and F_0, to leading order at $t \to t_c$, only the derivatives diverge as a power law:

$$\frac{\partial^3 F_0}{\partial t^3} = \frac{1}{2\, t_c}\,(1 - t/t_c)^{-1/2} + o((1 - t/t_c)^{-1/2})$$
$$\frac{\partial F_1}{\partial t} = \frac{1}{24\, t_c}\,(1 - t/t_c)^{-1} + o((1 - t/t_c)^{-1})$$

Let us compute $2\,u_g =$ singular part of $\partial^2 F_g/\partial t^2$, we have

$$u_0 = -\frac{1}{2} \quad , \quad u_1 = \frac{1}{48\, t_c^2} \quad , \quad u_2 = \frac{49}{3^2 * 2^8\, t_c^4} \quad , \quad \ldots$$

$$\text{for } g \geq 2\,, \qquad u_g = \frac{\tilde{F}_g}{t_c^{2g}}\,\frac{5}{4}\,(2-2g)\,(\frac{5}{4}(2-2g) - 1),$$

and define the formal series

$$u(s) = \sum_g u_g t_c^{2g}\, s^{(1-5g)/2} = -\frac{1}{2}\, s^{1/2} + \frac{1}{48}\, s^{-2} + \frac{49}{3^2 * 2^8}\, s^{-9/2} + \ldots$$

The values which we have found for u_0, u_1, u_2 indicate that $u(s)$ seems to satisfy the Painlevé I equation to the first few orders

$$3u^2 + u''/2 = \frac{3}{4}\, s + O(s^{-13/2}).$$

Our goal in this chapter, is to prove that indeed $u(s)$ satisfies Painlevé I equation to all orders:

$$3u^2 + u''/2 = \frac{3}{4}\, s.$$

This Painlevé equation determines all the coefficients u_g, and thus $\tilde{F}_g$, i.e. it gives the asymptotic numbers of large maps.

The Liouville minimal model of conformal field theory coupled to quantum gravity, predicts that the generating function of "number of surfaces", should satisfy the Painlevé I equation, so what we find is an agreement between the asymptotic number of large maps, and the Liouville conformal field theory of gravity.

Mesh size

The average number of quadrangles is $< n_4 >= t_4 \frac{\partial \ln F_g}{\partial t_4}$, and thus, if we say that all quadrangles have the same area ϵ^2 (we call mesh size the side of each quadrangle, that is ϵ), the average area is:

$$< \text{Area} >= \epsilon^2\, < n_4 > \sim \frac{5}{4}(2-2g)\,\frac{\epsilon^2}{\frac{t}{t_c} - 1}$$

If we want to have a good continuous limit of random surfaces, we require the area to remain finite, and it means that we should choose:

$$\epsilon^2 \sim t_c - t$$

Therefore, the distance to critical point $t_c - t$ can be interpreted as the mesh area, i.e. the area of elementary quadrangles.

1.3 About double scaling limits and Liouville quantum gravity

Origin of the name "double scaling limit"

Remember that we have defined $\ln Z = \sum_g N^{2-2g} F_g$, where Z is the generating function of all maps of all genus not necessarily connected. Anticipating on theorem 3.1, we notice that $F_g \sim \tilde{F}_g\, t_c^{2-2g}\, (1-t/t_c)^{(2-2g)\mu}$ with the exponent of $(1-t/t_c)$ proportional to $2-2g$. Thus, it is possible to define a rescaled parameter $\tilde{N} = N\, t_c\, (1-t/t_c)^{\mu}$, and a series:

$$\ln \tilde{Z} = \sum_{g=0}^{\infty} \tilde{N}^{2-2g}\, \tilde{F}_g$$

such that $\tilde{Z}$ is the "limit" of Z, in the "double scaling limit" (double because we take a limit on both N and t):

$$\begin{cases} t \to t_c \\ N \to \infty \end{cases} \qquad N\, t_c\, (1-t/t_c)^{\mu} = \tilde{N} = \text{finite} \qquad \longrightarrow \;\; Z \sim \tilde{Z}.$$

This double scaling limit $\tilde{Z}$ is to be viewed as the generating series of the continuous limit of maps.

From large maps to Liouville gravity

$\tilde{F}_g$ is the generating function of asymptotic numbers of large maps of genus g, rescaled by a power of the mesh size.

In a similar manner, one is also interested in the double scaling limits of $W_n^{(g)}$'s counting asymptotic numbers of large maps of genus g with n asymptotically large marked faces.

The guess made by physicists working in quantum gravity in the 80's and 90's, was that those double scaling limit generating functions $\tilde{F}_g$ and $\tilde{W}_n^{(g)}$, should coincide with correlation functions of Liouville conformal field theory coupled to gravity. This guess was supported by heuristic asymptotics of convergent matrix integrals, hoped to be valid for formal integrals.

On the conformal field theory side, due to conformal invariance, the correlation functions of a conformal field theory, must have the symmetry of some representations of the conformal group, that is they are given in terms of representations of the Virasoro algebra.

Finite representations of the conformal group were classified (in the famous Kacs table [27]) and are called minimal models, they are labeled by 2 integers (p, q). For the minimal models, the partial differential equations imply that the partition function has to satisfy a non-linear ordinary differential equation. For example, the minimal model $(3, 2)$ is called pure gravity, and its generating function satisfies the Painlevé I equation.

The minimal models are also related to finite reductions of the KP (Kadamtsev-Petiashvili) integrable hierarchy.

If the asymptotics generating functions $\tilde{F}_g$ of large maps were related to Liouville gravity, that would mean that $\tilde{Z}$ would be a tau-function for the KP (Kadamtsev-Petviashvili) hierarchy of integrable equations, and in particular $\tilde{Z}$ should satisfy some non-linear differential equations with the Painlevé property. We shall derive these differential equations below in section 4.

Thus, in principle, if we want to compare large maps to Liouville quantum gravity, we have to check that the generating function of the $\tilde{F}_g$ and $\tilde{W}_n^{(g)}$'s, satisfy the differential equations of some (p, q) minimal model. In particular, we have to check that $\tilde{Z}$ is indeed the tau-function of a minimal model reduction of the KP hierarchy

$$\tilde{Z} \stackrel{?}{=} \mathrm{Tau-function\,of}\,(p,q)\,\mathrm{reduction\,of\,KP\,hierarchy}.$$

We also have to check that the scaling exponents of large maps, are those computed by KPZ (Khniznik Polyakov Zamolodchikov) [56]

$$\mathrm{KPZ\ \ exponent}\ \ \gamma = \frac{-2}{p+q-1}\,, \qquad F_g \stackrel{?}{\sim} \tilde{F}_g\ \ (1-t/t_c)^{(2-2g)(1-\gamma/2)}.$$

All this was done at a heuristic level by physicists in the 90's. We provide a mathematical proof below in this chapter.

2 Critical spectral curve

Here we study what special happens at $t = t_c$? Why generating functions diverge ?

2.1 Spectral curves with cusps

In chapter III, we have seen that the F_g's for $g \geq 2$ are rational fractions of α and γ^2 (F_0 and F_1 also contain logarithms of rational fractions of α and γ^2). α and γ themselves are obtained by solving an algebraic equation, and thus they may have singularities. One can compute (see theorem 4.5 section III.4.3):

$$\frac{d\gamma}{dt} = \frac{1}{4}\left(\frac{1}{y'(1)} + \frac{1}{y'(-1)}\right)$$

and $y'(1)$ and $y'(-1)$ are themselves algebraic functions of t. Therefore we see that γ is singular whenever $y'(1) = 0$ or $y'(-1) = 0$. Without loss of generality, let us consider that $y'(1)$ vanishes at $t = t_c$.

We are thus led to study the behavior of $y(z)$ near $z = 1$. For any t, let us compute the Taylor expansion of $x(z)$ and $y(z)$ at $z = 1$.

Since $x(z) = \alpha + \gamma(z + 1/z)$ we always have $x'(1) = 0$, and to the order $(z-1)^2$ we have

$$x(z) \sim x(1) + \gamma(z-1)^2 + O((z-1)^3),$$

and thus

$$z - 1 \sim \sqrt{\frac{x-a}{\gamma}}.$$

And $y(z) \sim (z-1)y'(1) + \frac{1}{2}(z-1)^2 y''(1) + \frac{1}{6}(z-1)^3 y'''(1) + \ldots$. Generically y behaves like a square root near its branchpoints:

$$y \sim y'(1)\sqrt{\frac{x-a}{\gamma}} + O((x-a)^{\frac{3}{2}})$$

At $t = t_c$, however, since $y'(1)$ vanishes, y no longer behaves as a square root, it has a cusp singularity of the form $y \sim (x-a)^{3/2}$, and if more derivatives of y vanish, it has a cusp singularity of the form:

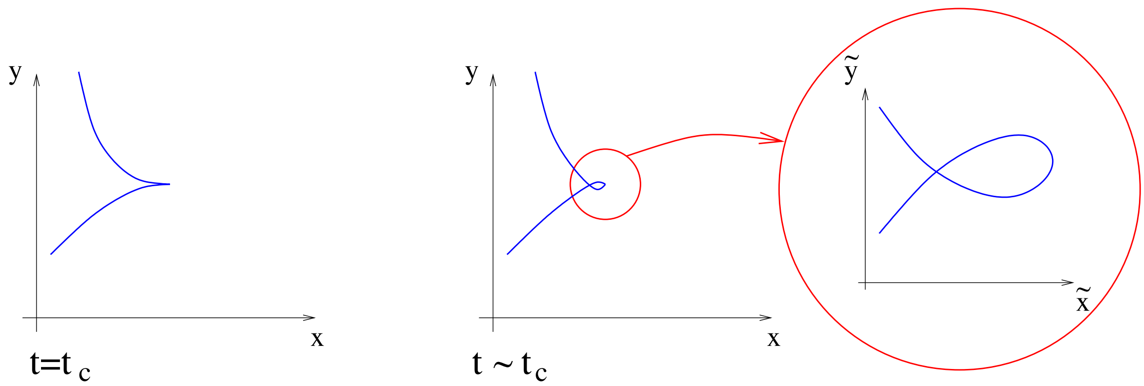
$$y \sim (x-a)^{p/q}.$$

Here, for maps, y is always the square root of some polynomial, so that p/q must be half–integer, i.e. $q = 2$ and $p = 2m+1$ where m corresponds to the first non-vanishing derivative of y at $z = 1$, that is $y(z) \sim O((z-1)^{2m+1})$.

Remark 2.1 In more general maps, for instance colored maps carrying an Ising model (see chapter VIII), or a $O(\mathfrak{n})$ model, other exponents p/q are possible. The Ising model allows to reach any rational p/q singularity. The $O(\mathfrak{n})$ model allows to reach all p/q singularities (not necessarily rational) such that $\mathfrak{n} = -2\cos(\frac{p}{q}\pi)$.

The integers p and q are going to be related to the (p, q) minimal model.

If t is close to t_c, the curve $y(x)$ is not singular, but it approaches a singularity. So, let us zoom into a small region near the branchpoint.



For example, consider that the branchpoint which becomes singular is the one at $z = 1$ (in case both branchpoints become singular there are extra factors of 2 in some formulae, this is the case for even maps).

Example: quadrangulations

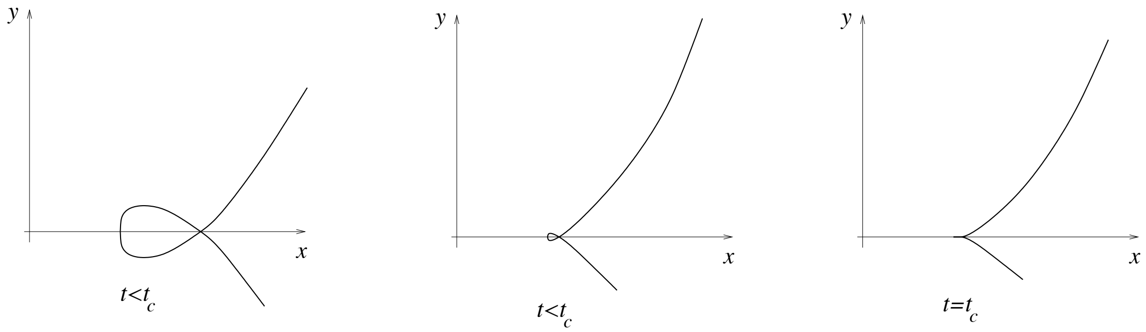
If one plots the spectral curve y versus x, one sees that at $t \neq t_c$, the curve (x,y) is regular, it behaves generically like a square root near its branch points $x = \pm 2\gamma$, it has everywhere a tangent (at the branchpoints the tangent is vertical). At $t = t_c = \frac{1}{12t_4}$: the curve (x,y) ceases to be regular, it has a cusp singularity, it has no tangent at $z = 1$. Indeed, we have (from eq.(III-1-15)):

$$y = -\frac{t_4}{2}\,(x^2 - 4\gamma^2 + 3\gamma^2\frac{\gamma^2 - 2t}{\gamma^2 - t})\,\sqrt{x^2 - 4\gamma^2} \qquad , \qquad \gamma^2 = \frac{1 - \sqrt{1 - 12tt_4}}{6t_4}$$

At $t = t_c = 1/12t_4$ we have $\gamma^2 = 2t$ and thus:

$$t = t_c \quad \Rightarrow \qquad y = -\frac{t_4}{2}\,(x^2 - 8t)^{3/2}$$

At $t = t_c$, the square root singularity at $x = 2\gamma$ is replaced by a power $3/2$ singularity.



In a vicinity of the critical point, we parametrize t_4 as:

$$tt_4 = \frac{1 - \epsilon^2}{12}$$

where ϵ is the "mesh size".

In the small ϵ limit we have the Taylor expansion:

$$\gamma^2 \sim 2t\,(1 - \epsilon) \quad + O(\epsilon^2)$$

and if we rescale x in a vicinity of the branch-point $x \sim 2\gamma$ as:

$$x = \sqrt{8t}\,(1 + \frac{1}{4}\epsilon(\zeta^2 - 2))$$

we find that y behaves like:

$$y \sim -\frac{\sqrt{t}}{3}\,\epsilon^{\frac{3}{2}}\,(\zeta^3 - 3\zeta) \quad + O(\epsilon^{\frac{5}{2}})$$

This corresponds to having rescaled the Zhukovsky's variable near $z = 1$ as

$$z = 1 + \sqrt{\frac{\epsilon}{2}}\,\zeta + O(\epsilon).$$

Let us define the parametric curve $(\tilde{x}, \tilde{y})$ defined by keeping only the leading non–trivial behaviors of x and y at small ϵ:

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2 \\ \tilde{y}(\zeta) = \zeta^3 - 3\zeta \end{cases}$$

it is called the "blow up" of the curve (x, y) near its singularity.

This blown up curve is going to play an important role below.

2.2 Multicritical points

The previous example of just quadrangulations is in some way too simple, as it does not contain any "multicritical point". The reason is that it depends only on 1 variable tt_4.

Example: quadrangles + hexagons

In order to illustrate a more general type of mutlicritical behaviour, consider maps with both quadrangles (weighted by t_4), and hexagons (weighted by t_6), in particular they are even. We have:

$$V'(x) = x - t_4 x^3 - t_6 x^5$$

The spectral curve is easily computed with theorem III.1.1:

$$y = \frac{1}{2}\left(t_6(x^2 - 4\gamma^2)^2 + (t_4 + 10t_6\gamma^2)(x^2 - 4\gamma^2) + 3t_4\gamma^2 + 20t_6\gamma^4 - \frac{t}{\gamma^2}\right)\sqrt{x^2 - 4\gamma^2}$$

where γ^2 is the solution of the following algebraic equation, and which behaves like $t + O(t^2)$ at small t:

$$t = \gamma^2 - 3t_4\gamma^4 - 10t_6\gamma^6, \qquad \text{(V-2-1)}$$

i.e. according to chapter III

$$\gamma^2 = \sum_{k,l} t^{k+l+1} \frac{(2k+3l)!}{(k+2l+1)!\; k!\; l!}\; (3t_4)^k\; (10\, t_6)^l.$$

We have now 2-parameters t_4 and t_6. For each t_4, we can find a critical value of t_6 at which y has a cusp $y \sim (x - 2\gamma)^{3/2}$. It happens when $3t_4\gamma^2 + 20t_6\gamma^4 - \frac{t}{\gamma^2} = 0$, i.e.

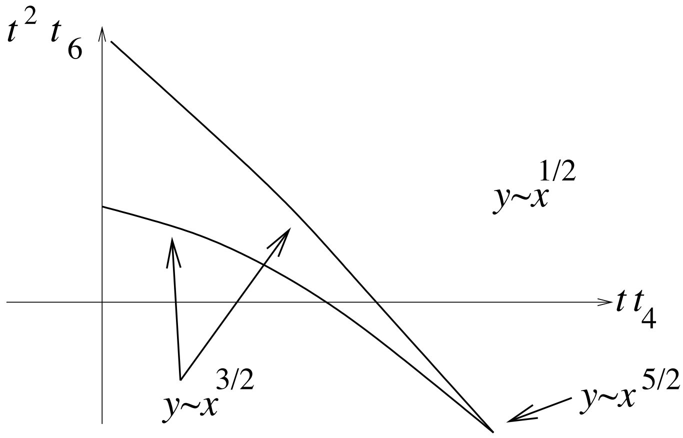
$$t_6 = \frac{2 - 27\, tt_4 \pm 2(1 - 9\, tt_4)^{\frac{3}{2}}}{270\; t^2} \qquad \text{(V-2-2)}$$

This gives two critical lines in the (t_4, t_6) plane.

Then, if in addition to eq.(V-2-2), we have $t_4 + 10t_6\gamma^2 = 0$, we can find a point (at the intersection of the two critical lines) where $y \sim (x - 2\gamma)^{5/2}$. This point is at

$$t_4 = \frac{1}{9\; t} \qquad , \qquad t_6 = -\frac{1}{270\; t^2}$$

This is best represented on a phase diagram:



Now, our goal is to consider t_4 and t_6 a little bit away from the critical point, and study the limit of generating functions of maps, as we approach the critical point.

Of course, depending on how we approach the critical point, we can find different asymptotic behaviors. The asymptotics for the F_g's are going to be different if we approach the critical point along a critical line, or from a generic direction.

Let us consider a small vicinity of the critical point, parametrized as:

$$t_4 = \frac{1}{9t}\,(1 - \epsilon^2\,\frac{\tilde{t}_0}{3}) \qquad , \qquad t_6 = -\frac{1}{270t^2}\,(1 - \epsilon^2\,\tilde{t}_0 + \epsilon^3\,s)$$

where ϵ is small (it is the mesh size), and $s, \tilde{t}_0$ are of order $O(1)$.

It will be more convenient to use a variable u_0 instead of s:

$$s = 8\,u_0^3 - 2\tilde{t}_0\,u_0.$$

In some sense u_0 measures the distance to critical point along the critical line, and $\tilde{t}_0 - 12u_0^2$ measures the "distance" transverse to the critical line.

The equation eq.(V-2-1) for γ gives:

$$\frac{3t}{\gamma^2} = 1 + 2\epsilon u_0$$

and if we rescale x in a vicinity of the branch-point 2γ as:

$$x \sim \sqrt{3t}\,(2 + \epsilon\,(\zeta^2 - 2u_0) + O(\epsilon^2))$$

we find that y behaves like:

$$y \sim \sqrt{\frac{t}{3}}\;\epsilon^{\frac{5}{2}}\;\left(-\frac{8\,\zeta^5}{5} + 8\,u_0\,\zeta^3 + (\tilde{t}_0 - 12\,u_0^2)\zeta\right)(1 + O(\epsilon))$$

The parametric curve $(\tilde{x}, \tilde{y})$

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2u_0 \\ \tilde{y}(\zeta) = -\frac{8}{5}\,(\zeta^5 - 5\,u_0\zeta^3 + \frac{15\,u_0^2}{2}\,\zeta) + \tilde{t}_0\,\zeta \end{cases}$$

is called the "blown up" of the curve (x,y) near its singularity. Again, anticipating on section 4, we notice that the exponents 5 and 2, are a hint that this spectral curve has to do with the $(5,2)$ minimal model.

The differential form ydx plays a key role in the recursive computations of $W_n^{(g)}$'s, and it scales like:

$$ydx \sim \, t\,\epsilon^{7/2}\,\, \tilde{y}\, d\tilde{x} + O(\epsilon^{9/2}).$$

Remark 2.2 If we would compare the formal matrix model for maps to a convergent matrix integral, then the large N limit of the density of eigenvalues would be $\rho(x)dx = \frac{N}{2\pi i t}\, y\, dx$. Thus, we see that if we choose

$$\epsilon \sim N^{-2/7},$$

then a region of size of order ϵ near the edge, contains a finite number of eigenvalues of the random matrix. This is a hint that the double scaling limit to be considered will be $t \to t_c$ and $N \to \infty$ and $N(1-t/t_c)^{7/4} = O(1)$.

Multicritical points, general case

More generally, when we consider maps, we have a spectral curve (x,y) depending on some parameters $t_3, t_4, \ldots t_d$ and t. As we have already noticed, the spectral curve depends only on the rescaled parameters $t^{\frac{k}{2}-1}\, t_k$, and the parameter t is redundant, but for further convenience we prefer to keep it.

In the space of parameters t_i, there exists critical sub-manifolds, corresponding to various singular behaviours for the spectral curves (x,y), of the form $y \sim (x-a)^{p/q}$, where $q=2$ and $p=2m+1$.

Consider a critical point $t_i = t_{ic}$, at which we have $y \sim (x-a)^{m+\frac{1}{2}}$.

When we move away from this point, we may move along various directions, for instance along a submanifold where $y \sim (x-a)^{m'+\frac{1}{2}}$ with $m' < m$, or we can also move into a non critical direction $m' = 0$.

Therefore, it is better to reparametrize our parameters t, t_i's as functions of more appropriate parameters $\epsilon, \tilde{t}_i$'s:

$$t_i = t_i(\epsilon, \tilde{t}_1, \ldots, \tilde{t}_m) \qquad \text{where} \quad \epsilon^2 = t_c - t$$

and in such a way that the spectral curve can be written in the regime $\epsilon \to 0$ and $\tilde{t}_i = O(1)$ as:

$$\left\{ \begin{array}{l} x(\zeta) \sim a_c + \gamma_c \epsilon(\zeta^2 - 2u) \quad + O(\epsilon^2) \\ y(\zeta) \sim \frac{t_c}{\gamma_c}\, \epsilon^{m+\frac{1}{2}} (\sum_{m'=0}^{m} \tilde{t}_{m'} Q_{m'}(\zeta)) \quad + O(\epsilon^{m+\frac{3}{2}}) \end{array} \right.$$

where

$$Q_{m'}(\zeta) = \sum_{j=0}^{m'} \frac{(-u)^j}{j!}\, \frac{(2m'+1)!!}{(2m'-2j+1)!!}\, \zeta^{2m'-2j+1} = \left((\zeta^2 - 2u)^{m'+\frac{1}{2}} \right)_+ \qquad \text{(V-2-3)}$$

is a polynomial of ζ of degree $2m'+1$ (it is the polynomial part of the large ζ Laurent series expansion of $(\zeta^2 - 2u)^{m'+\frac{1}{2}}$).

The first few are

$$Q_0(\zeta) = \zeta \qquad , \qquad Q_1(\zeta) = \zeta^3 - 3u\zeta, \qquad , \qquad Q_2(\zeta) = \zeta^5 - 5u\,\zeta^3 + \frac{15\,u^2}{2}\,\zeta$$

The spectral curve now depends on the parameters ϵ, u, and $\tilde{t}_i$, $i = 1, \dots, m$.

We have an extra parameter u, but we shall see below, that some consistency condition imply that u has to be a certain function of the $\tilde{t}_i$'s.

At $\epsilon \neq 0$, the spectral curve is regular, its branchpoints are of square root type. The curve becomes singular in the $\epsilon \to 0$ limit, and depending on the $\tilde{t}_i$'s, it may become critical or multicritical along some critical submanifolds.

Our goal is to study how the F_g's diverge in the limit $\epsilon \to 0$ (i.e. $t - t_c \to 0$). We are going to prove in theorem 3.1 below, that (remember that $\epsilon^2 = t_c - t$):

$$F_g \sim (1 - t/t_c)^{(2-2g)\mu}\; t_c^{2-2g}\; \tilde{F}_g(\tilde{t}_i) \quad (1 + o(1))$$

the scaling exponent $\mu = \frac{2m+3}{2m+2}$, and the values of $\tilde{F}_g$ are computed in theorem 3.1 below, and we shall find that the coefficients $\tilde{F}_g$ are the symplectic invariants (see chapter VII) of the blown up spectral curve:

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2u \\ \tilde{y}(\zeta) = \sum_{m'=0}^{m} \tilde{t}_{m'} Q_{m'}(\zeta). \end{cases}$$

Then, we shall show that the symplectic invariants of that curve, are related to the $(2m+1, 2)$ minimal model, and their generating function satisfies the $(m+1)^{\text{th}}$ Painlevé I equation.

3 Computation of the asymptotic $W_n^{(g)}$'s

Here, we compute how the functions $W_n^{(g)}(x_1, \dots, x_n)$ behave in a small region of size δ around a branchpoint ($z = +1$ for instance). We shall study this behavior independently of being close to a critical point or not, i.e. whether the curve behaves like a square root $y \sim \sqrt{x-a}$ or like any other power $y \sim (x-a)^{p/q}$.

Also here, we choose a small size δ on the spectral curve (i.e. in the z variable), independently of any mesh size ϵ. It is only later that we shall relate the two.

We thus rescale the Zhukovsky variables z_i's

$$z_i = 1 + \delta\,\zeta_i$$

and thus $x_i = x(z_i) = \alpha + \gamma(z_i + 1/z_i)$ gives:

$$x_i = x(1) + \gamma\,\delta^2 \zeta_i^2 + O(\delta^3).$$

Our goal is to study the asymptotic behavior of $W_n^{(g)}(x_1, \dots, x_n)$ in the limit $\delta \to 0$.

For latter purposes, we will also be interested in situations where the size δ may depend (or not depend) on the times t, t_k, and thus $x(1)$ and γ may also have a small δ expansion.

For example, if we are near a critical point, we may want to choose the scale δ of the form $\delta \sim (t_c - t)^\nu$ with some appropriate exponent ν ($\nu = 0$ if δ is independent of t).

However, for the moment, we do not assume any particular relationship, in fact we allow any arbitrary relationship. Thus we find, by doing a Taylor expansion in powers of δ:

$$\left\{ \begin{array}{l} x(z) \sim x(1) + \gamma\, \delta^q \tilde{x}(\zeta) + o(\delta^2) \\ y(z) \sim \frac{t}{\gamma}\, \delta^p \tilde{y}(\zeta) + o(\delta^p) \end{array} \right. \qquad , \qquad \tilde{x}(\zeta) = \zeta^2 - 2u \quad , \quad q = 2$$

where p is the leading exponent in powers of δ, and $\tilde{y}$ is, for the moment, an almost arbitrary function of ζ. For example, if we assume that y would behave locally like $(x-a)^{p/q}$ then $\tilde{y}(\zeta)$ would be a polynomial of ζ of degree p.

The coefficient u comes from the $O(\delta^2)$ term in the expansion of $x(1) = x_0 + x_1\delta - 2\gamma\, u\delta^2 + O(\delta^3)$, it is related to the choice of relationship between δ and t, t_i's, and this choice will depend on the kind of critical point under consideration.

We call the curve $\tilde{y}(\tilde{x})$ the blown up of the curve $y(x)$ in the region of size δ:

$$\left\{ \begin{array}{l} \tilde{x}(\zeta) \\ \tilde{y}(\zeta) \end{array} \right. .$$

All the generating functions F_g and $W_n^{(g)}$ are given by theorem III.3.1 and theorem III.4.3, i.e. by residue formulae in the vicinity of $z = \pm 1$. Near $z = +1$, we write $z = 1 + \delta\zeta$, and near $z = -1$, we have $z + 1 = 2 + O(\delta)$. Let us study how each term behaves in the small δ limit. The fundamental second kind differential $B(z_0, z) = 1/(z_0 - z)^2$ behaves like:

$$B(z_0,z) \sim \begin{array}{|l|l|l|} \hline & z \text{ near } +1 & z \text{ near } -1 \\ \hline z_0 \text{ near } +1 & \delta^{-2}\, \tilde{B}(\zeta_0,\zeta) & O(1) \\ z_0 \text{ near } -1 & O(1) & O(1) \\ \hline \end{array} \times (1 + O(\delta)),$$

where $\tilde{B}(\zeta_0, \zeta)$ is the fundamental second kind differential of the curve $\tilde{y}(\tilde{x})$:

$$\tilde{B}(\zeta_0,\zeta) = \frac{1}{(\zeta - \zeta_0)^2}.$$

Similarly, the kernel K (see eq.(III-7-1) in chapter III)

$$K(z_0,z) = \frac{1}{2}\left(\frac{1}{z_0 - z} - \frac{1}{z_0 - \frac{1}{z}}\right) \frac{1}{2y(z)\, x'(1/z)}$$

behaves like:

$$K(z_0,z) \sim \begin{array}{|l|l|l|} \hline & z \text{ near } +1 & z \text{ near } -1 \\ \hline z_0 \text{ near } +1 & \frac{1}{t}\, \delta^{-(p+q)} \tilde{K}(\zeta_0,\zeta) & O(1) \\ z_0 \text{ near } -1 & O(\delta^{-(p+q-1)}) & O(1) \\ \hline \end{array} \times (1 + O(\delta)),$$

where $\tilde{K}(\zeta_0,\zeta)$ is the recursion kernel (see chapter VII) of the spectral curve $(\tilde{x},\tilde{y})$:

$$\tilde{K}(\zeta_0,\zeta) = \frac{1}{2}\left(\frac{1}{\zeta_0-\zeta} - \frac{1}{\zeta_0+\zeta}\right)\,\frac{1}{2\tilde{y}(\zeta)\,\tilde{x}'(\zeta)}.$$

Therefore, we see that the leading contribution to $\omega_{n+1}^{(g)}(1+\delta\zeta_0,\ldots,1+\delta\zeta_n)$ is given by the case where all residues are taken near $+1$, and can be computed only in terms of $\tilde{B}$ and $\tilde{K}$. By an easy recursion on $2g+n-2$, we obtain:

Theorem 3.1 *Double scaling limits of correlation functions*

$$\omega_n^{(g)}(1+\delta\zeta_1,\ldots,1+\delta\zeta_n) \sim t^{2-2g-n}\,\delta^{(2-2g-n)(p+q)}\,\delta^{-n}\;\tilde{\omega}_n^{(g)}(\zeta_1,\ldots,\zeta_n)\;\left(1+O(\delta)\right)$$

and $\tilde{\omega}_n^{(g)}$ *are determined by the recursion relation:*

$$\tilde{\omega}_2^{(0)}(\zeta_1,\zeta_2) = \frac{1}{(\zeta_1-\zeta_2)^2}$$

$$\tilde{\omega}_{n+1}^{(g)}(\zeta_0,J) = \mathop{\mathrm{Res}}_{\zeta\to 0}\;\tilde{K}(\zeta_0,\zeta)\left[\tilde{\omega}_{n+2}^{(g-1)}(\zeta,-\zeta,J) + \sum_{h=0}^{g}\sum_{I\subset J}\tilde{\omega}_{1+|I|}^{(h)}(\zeta,I)\tilde{\omega}_{1+n-|I|}^{(g-h)}(-\zeta,J/I)\right] \qquad \text{(V-3-1)}$$

where

$$\tilde{K}(\zeta_0,\zeta) = \frac{1}{2}\left(\frac{1}{\zeta_0-\zeta} - \frac{1}{\zeta_0+\zeta}\right)\,\frac{1}{(\tilde{y}(\zeta)-\tilde{y}(-\zeta))\tilde{x}'(-\zeta)}.$$

Therefore, we have found the scaling limit of $W_n^{(g)}$ in a small region of size δ.

Remark 3.1 Notice that the recursion relation eq.(V-3-1) for the $\tilde{\omega}_n^{(g)}$'s, is very similar to the recursion relation of theorem 3.1 for the $\omega_n^{(g)}$'s themselves. In fact both are special cases of the general "Topological recursion" introduced in [37], which is presented in chapter VII in this book. In some sense, the topological recursion commutes with taking limits.

Then, one could be tempted to apply the same method to the computation of F_g (with $g\geq 2$), from theorem III.4.3:

$$(2-2g)\,F_g = \mathop{\mathrm{Res}}_{z\to +1}\Phi(z)\omega_1^{(g)}(z)dz + \mathop{\mathrm{Res}}_{z\to -1}\Phi(z)\omega_1^{(g)}(z)dz \qquad \text{(V-3-2)}$$

Indeed, we have seen that $\omega_1^{(g)}(1+\delta\zeta) \sim \delta^{(1-2g)(p+q)-1}\,\tilde{\omega}_1^{(g)}(\zeta)$, whereas near $z=-1$ (if $z=-1$ is not critical) we typically have $\omega_1^{(g)}(z) = o(\delta^{(1-2g)(p+q)-1})$. Thus, naively, one is tempted to write that the leading behavior of F_g would be:

$$F_g \sim \delta^{(2-2g)(p+q)}\,t^{2-2g}\,\tilde{F}_g \quad (1+o(1))$$

where

$$\tilde{F}_g = \frac{1}{2-2g}\mathop{\mathrm{Res}}_{\zeta\to 0}\tilde{\Phi}(\zeta)\tilde{\omega}_1^{(g)}(\zeta)d\zeta$$

with $\tilde{\Phi}'(\zeta) = \tilde{y}(\zeta)\tilde{x}'(\zeta)$.

However, this formula can be valid only if $\tilde{F}_g \neq 0$, otherwise this means that in fact F_g is given by subdominant contributions and all what we get is in that case

$$\tilde{F}_g = 0 \qquad \Leftrightarrow \qquad F_g = o(\delta^{(2-2g)(p+q)}).$$

This is not surprising, because F_g is not a function of δ, it is a function of the t_i's and so far we have not considered the relationship between δ and the t_i's. For instance if one chooses δ independent of the t_i's, then in that case F_g should clearly not depend on δ.

Remark 3.2 In case where both $z = -1$ and $z = +1$ are critical points of the curve (x, y), it may happen that the two terms of eq.(V-3-2) are of the same order of magnitude.

For instance this is the case for even maps, where all functions $\omega_n^{(g)}$ have a symmetry $z \to -z$, and in that case, we get an overall prefactor 2:

$$F_g \sim 2\, \delta^{(2-2g)(p+q)}\, t^{2-2g}\, \tilde{F}_g \quad (1 + O(\delta))\,.$$

3.1 Double scaling limit of F_g

In the case of the spectral curve (x, y) of the enumeration of maps, which has near a critical point a cusp singularity of type $y \sim (x-a)^{p/q}$ (with $p = 2m+1$, $q = 2$) near its branchpoint, we choose a scale $\delta = (1 - t/t_c)^\nu$, and the blow up is of the form

$$\left\{ \begin{array}{lll} x(z) \sim x(1) + \delta^q\, \gamma(t_c)\tilde{x}(\zeta) + o(\delta^q) & , & \deg \tilde{x} = q = 2 \\ y(z) \sim \delta^p \frac{t_c}{\gamma(t_c)}\, \tilde{y}(\zeta) + o(\delta^p) & , & \deg \tilde{y} = p = 2m+1 \end{array} \right.$$

where $\tilde{y}(\zeta)$ is a polynomial of ζ of degree p. We parametrize the blown up spectral curve as:

$$\left\{ \begin{array}{l} \tilde{x}(\zeta) = \zeta^2 - 2u \\ \tilde{y}(\zeta) = \sum_{k=0}^m \tilde{t}_k Q_k(\zeta) \end{array} \right.$$

where we decompose the polynomial $\tilde{y}(\zeta)$ onto the basis of the Q_k's defined in eq.(V-2-3).

Moreover, we choose the t_k close to their critical value, and we define the $\tilde{t}_j$ to be the distance from the critical point, measured in eigendirections, i.e. in the form:

$$t_k = t_{k,c} + \sum_j C_{k,j}\, \delta^{\nu_j}\, \tilde{t}_j.$$

It can thus also be written $t_k = t_{k,c} + \sum_j C_{k,j}\, (1 - t/t_c)^{\nu\nu_j}\, \tilde{t}_j$. The j^{th} exponent $\nu\nu_j$ is called the "dressed exponents", of the flow which moves from the $(2m+1, 2)$ singularity to the $(2j+1, 2)$ singularity (indeed $\tilde{t}_j$ is associated to $Q_j(\zeta)$):

$$\text{dressed exponents} \qquad \nu\nu_j.$$

It remains to determine the exponents ν and ν_j (and check that they match with the KPZ formula [56]).

In this purpose, we recall lemma III. 1.4, we have (at fixed t_k):

$$\frac{\partial x(z)}{\partial z}\,\frac{\partial y(z)}{\partial t} - \frac{\partial y(z)}{\partial z}\,\frac{\partial x(z)}{\partial t} = \frac{1}{z}.$$

which can be rewritten, in the regime $z = 1 + \delta\,\zeta$, and $\delta \sim (1 - t/t_c)^\nu$, as:

$$\sum_k \tilde{t}_k((p-\nu_k)\,\tilde{x}'(\zeta)\,Q_k(\zeta) - q\,Q_k'(\zeta)\,\tilde{x}(\zeta)) = \frac{-1}{\nu}\,\delta^{\frac{1}{\nu}-(p+q-1)}\,(1+o(1)). \qquad \text{(V-3-3)}$$

From their definition (see eq.(V-2-3)), one sees that the Q_k satisfy

$$(2k+1)\tilde{x}'Q_k - 2\tilde{x}Q_k' = -2(2k+3)(-u/2)^{k+1}\,\frac{(2k+2)!}{(k+1)!\,(k+1)!}.$$

Since the Q_k form a basis of odd polynomials of degree $\leq 2m+1$, the only possibility for the right-hand-side of eq.(V-3-3) to be a constant, is to choose $p - \nu_k = 2k+1$.

Also, since the left-hand-side of eq.(V-3-3) is independent of δ, we must have $1/\nu = p+q-1$:

$$\nu = \frac{1}{p+q-1} \qquad , \qquad \nu_k = p - (2k+1) = 2(m-k).$$

We also find that u is solution of a polynomial equation:

$$\sum_k \tilde{t}_k\,\frac{(2k+3)!}{(k+1)!^2}\,(-u/2)^k = \frac{p+q-1}{2}. \qquad \text{(V-3-4)}$$

Therefore, the generating functions of large maps are asymptotically given by

Theorem 3.2 *Double scaling limit of the F_g's enumerating functions of maps, at a (p,q) critical point ($p = 2m+1$, $q = 2$), for $g \geq 2$:*

$$F_g \sim (1 - t/t_c)^{(2-2g)\frac{p+q}{p+q-1}}\,t_c^{2-2g}\,\tilde{F}_g \quad + O((1-t/t_c)^{\nu+(2-2g)\frac{p+q}{p+q-1}})$$

where

$$\tilde{F}_g = \frac{C}{2-2g}\,\mathop{\mathrm{Res}}_{\zeta\to 0}\,\tilde{\Phi}(\zeta)\,\tilde{\omega}_1^{(g)}(\zeta) \qquad \text{(V-3-5)}$$

and where

$$\tilde{\Phi}'(\zeta) = \tilde{y}(\zeta)\tilde{x}'(\zeta)$$

and where generically $C = 1$. For cases where the 2 branchpoints are critical, we may have $C \neq 1$, in particular for even maps we have $C = 2$.

Therefore, we have computed the double scaling limit $\tilde{F}_g$ of F_g.

Remark 3.3 If $p = 1, q = 2$, i.e. if the spectral curve has a regular branchpoint $y \sim \sqrt{x-a}$, the Blown up spectral curve is simply $\tilde{y} = \sqrt{\tilde{x} + 2u}$, and one may check that this spectral curve has $\tilde{F}_g = 0$, which is expected since F_g is not divergent when the spectral curve is regular. In that case, F_g is given by the subdominant contributions. Therefore theorem 3.2 is useful only when $p \geq 3$.

Remark 3.4 The recursion relations eq.V-3-1 and eq.V-3-5 are very similar to the ones for $W_n^{(g)}$ and F_g of theorem.3.1 and theorem.4.3 in chapter III. We will show in chapter VII, that it is possible to define a common framework for both F_g and its double scaling limit $\tilde{F}_g$, namely the notion of a family of "symplectic invariants" attached to any spectral curve $y(x)$. The counting functions of maps as well as their scaling limits are special cases of those invariants.

In other words, if F_g is the g^{th} symplectic invariant of the spectral curve $y(x)$, then:

Theorem 3.3 *$\tilde{F}_g$ is the g^{th} symplectic invariant of the blown up spectral curve $\tilde{y}(\tilde{x})$.*

The notion of symplectic invariants of a spectral curve is explained in chapter VII.

3.2 Critical exponents and KPZ

In this subsection, we mention very briefly the link to KPZ. Readers can easily skip to the next section. We just sketch without details the link to conformal field theory, and refer the readers to reference books and reviews on the subject [27, 42].

Definition 3.1 *The critical exponents in quantum gravity are defined as:*

• *The "**string susceptibility exponent**" γ (often denoted γ_{string} in the physics literature) is such that $\gamma = \gamma_0$ and γ_g are related to how the generating function F_g (generating function for genus g surfaces) diverges when the mesh size $(1 - t/t_c)$ tends to 0 (or equivalentely, how it diverges at large area):*

$$F_0 \sim (1 - t/t_c)^{2-\gamma}\, t_c^2\, \tilde{F}_0 + \text{regular}$$

and for higher genus

$$F_g \sim (1 - t/t_c)^{2-\gamma_g}\, t_c^{2-2g}\, \tilde{F}_g.$$

• *The "**dressing exponents**" $\Delta_{j,1}$ are related to the scaling behaviors when one moves away from the $(2m+1, 2)$ critical point along a critical submanifold of codimension r (i.e. a $(2r+1, 2)$ critical submanifold), measured in mesh size, and normalized so that $\Delta_{1,1} = 0$ for $j = 1$. In other words it is related to the scalings*

$$t_k = t_{k,c} + \sum_j C_{k,j}\,(1 - t/t_c)^{\frac{\Delta_{j,1}-\Delta_{m,1}}{1-\Delta_{m,1}}}\,\tilde{t}_j.$$

We have thus proved that

Theorem 3.4 *The critical exponents are:*

$$2 - \gamma_g = (2 - 2g)\,(p+q)\nu = (2 - 2g)\,\frac{p+q}{p+q-1}$$

In particular in genus 0:

$$\gamma = \frac{-2}{p+q-1}.$$

The exponents $\Delta_{j,1}$ are related to $\nu_j = p - (2j+1) = 2(m-j)$ by:

$$\frac{\Delta_{j,1} - \Delta_{m,1}}{1 - \Delta_{m,1}} = \nu \nu_j = \frac{2(m-j)}{p+q-1}$$

and since $\Delta_{1,1} = 0$:

$$\Delta_{j,1} = \frac{2-2j}{4} = \frac{|p-qj| - |p-q|}{p+q-|p-q|}$$

They are those predicted by the Kac's table [27] and the KPZ formula [56].

Kac's table

We refer the reader to literature on Conformal Field Theory, for example [27].

Finite representations of the conformal group in 2 dimensions, are classified as the (p,q) minimal models. The (p,q) minimal model has central charge

$$c = 1 - 6\,\frac{(p-q)^2}{pq} = 1 - 6\left(\frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}\right)^2,$$

where we introduced the parameter $\kappa = \frac{4q}{p}$. This parameter κ is the one that appears in the famous SLE_κ processes, see the literature [29, 77].

Minimal models have a finite number of possible heighest weights. For the (p,q) minimal model The heighest weights of the (p,q) minimal models are labeled by two integers (r,s) with $0 < r < p$ and $0 < s < q$, and with the identification $(r,s) \equiv (p-r, q-s)$. Their heighest weights are given by the famous Kac's formula:

$$h_{r,s} = \frac{(ps-qr)^2 - (p-q)^2}{4pq}.$$

The weights $h_{r,s}$ are the exponents that control how the corresponding fields change under dilatations.

* The field $(1,1)$ has weight 0, it is called the "identity operator":

$$(1,1) \text{ field } = \text{Identity} \qquad , \qquad h_{1,1} = 0.$$

* The value of (r,s) which gives the minimum of $|ps-qr|$, is called the "most relevant operator", it has the smallest weight $h_{r,s}$.

* The unitary minimal models are those for which $|p-q| = 1$, and for them, the "most relevant operator" is the Identity $(1,1)$.

• Case $(p,q) = (2m+1, 2)$.

In that case, the central charge is

$$c = 1 - 3\,\frac{(2m-1)^2}{2m+1}.$$

There are m heighest weights correponding to $s = 1$ and $1 \leq r \leq m$, their weights are

$$h_{r,1} = \frac{(r-1)\,(r-2m)}{2(2m+1)}.$$

In that case, the most relevant operator is $(r, s) = (m, 1)$, its weight is:

$$h_{m,1} = \frac{-m\,(m-1)}{2(2m+1)}.$$

The only unitary models among the $(2m+1, 2)$ models, are the $(3, 2)$ model (pure gravity), with central charge $c = 0$, and the $(1, 2)$ model (Airy model) with central charge $c = -2$.

KPZ

Polyakov understood in 1981 [74], that conformal Field theories can be coupled to gravity, in a way preserving conformal invariance, by adding a new field: the Liouville field.

The Liouville field is constructed from the Gaussian free field, see [29], and was recently constructed in probability theory [24].

There are also exponents controling how the fields change with a dilatation, however, the coupling to gravity means that the metric itself changes under dilatations, and thus the exponents get "dressed" by gravity.

It is cutomary to measure the behavior under dilatations by measuring how the fields scale in powers of the area of the surface when the area becomes large, or equivalently how they scale in powers of the mesh size at small mesh.

Recall that for us the mesh size is $(1 - t/t_c)$.

The exponent γ_g controls the scaling of the partition function of genus g. In Liouville theory, the topology enters only through the integral of the curvature, which is proportional to the Euler characteristics $\chi = 2 - 2g$, and thus γ_g is expected to be a polynomial of degree 1 of the genus. We write it:

$$2 - \gamma_g = (1 - g)(2 - \gamma)$$

with $\gamma = \gamma_0$. In other words, the exponent γ should be such that

$$F_g \sim (1 - t/t_c)^{2-\gamma_g}\; t_c^{2-2g}\, \tilde{F}_g \sim (1 - t/t_c)^{(1-g)(2-\gamma)}\; t_c^{2-2g}\, \tilde{F}_g.$$

The KPZ formula, due to Knizhnik, Polyakov, Zamolodchikov [56], computes the dressing exponents $\Delta_{r,s}$ of the weights $h_{r,s}$. They claim that:

$$\frac{\kappa}{4}\,\Delta_{r,s}^2 + \left(1 - \frac{\kappa}{4}\right)\Delta_{r,s} = h_{r,s},$$

where $\kappa = \frac{4q}{p}$ is the SLE parameter.

For (p, q) minimal models, this gives:

$$\Delta_{r,s} = \frac{|ps - qr| - |p - q|}{p + q - |p - q|}.$$

Notice that the identity operator $(r, s) = (1, 1)$, is undressed:

$$\Delta_{1,1} = 0.$$

The most relevant operator $(m,1)$ has the dressing:

$$\Delta_{m,1} = \frac{1-|p-q|}{p+q-|p-q|}.$$

They also found the string exponent γ, associated to the most relevant operator (r,s):

$$\gamma_{r,s} = -\frac{2\,|ps-qr|}{p+q-|ps-qr|}.$$

KPZ formulae for the $(2m+1,2)$ minmal model

In that case we have:

$$\Delta_{r,s} = \frac{2-2r}{p+q-1} = \frac{1-r}{m+1},$$

and

$$\gamma = \gamma_{m,1} = -\frac{2}{p+q-1} = -\frac{1}{m+1}.$$

This is in agreement with our direct proof from the generating functions of maps.

3.3 Example: triangulations and pure gravity

Consider the generating function for triangulations. The potential is:

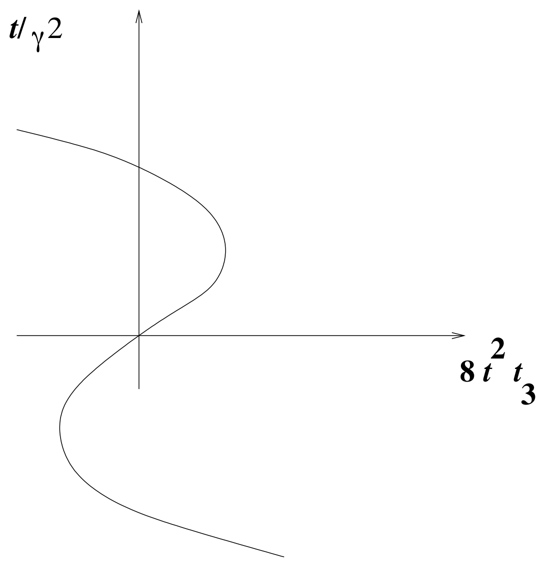
$$V(x) = \frac{x^2}{2} - t_3\frac{x^3}{3}$$

whose spectral curve was computed in section 1.8 of chapter III:

$$\left\{ \begin{array}{l} x(z) = \alpha + \gamma(z+1/z) \\ y(z) = \frac{1}{\gamma}(z-1/z) - t_3\gamma^2(z^2-z^{-2}) \end{array} \right.$$

where α, γ are determined by

$$r - r^3 = 8tt_3^2 \quad , \quad \gamma^2 = \frac{t}{r} \quad , \quad \alpha = \frac{1-r}{2t_3}.$$



The equation for γ becomes singular at $\sqrt{t}\, t_3 = t_c$, where

$$t_c = \frac{1}{2}\, 3^{-3/4} \qquad , \qquad r_c = \frac{1}{\sqrt{3}},$$

and one can check that at this point, the spectral curve has a (3/2) cusp $y \sim (x - x(1))^{3/2}$. This is the $(3,2)$ critical point, $p = 3 = 2m+1$ with $m = 1$, also called "pure gravity".

Near t_c we parametrize with a scaling δ:

$$\sqrt{t}\, t_3 = t_c(1 - \frac{3}{4}\delta^4),$$

so that we obtain

$$\gamma \sim \gamma(t_c)(1 - \frac{1}{2}\delta^2) + O(\delta^3) \qquad , \qquad \alpha \sim \alpha(t_c) - \gamma(t_c)\, \delta^2 + O(\delta^3)$$

where

$$\gamma(t_c) = 3^{1/4}\, \sqrt{t} \qquad , \qquad \alpha(t_c) = 3^{1/4} \sqrt{t}\, (\sqrt{3} - 1).$$

If we choose

$$z = 1 + \delta\, \zeta$$

we have:

$$\begin{cases} x(z) \sim \alpha(t_c) + 2\gamma(t_c) + 3^{1/4}\sqrt{t}\, \delta^2\, (\zeta^2 - 2) + o(\delta^2) \\ y(z) \sim \frac{\sqrt{t}}{3^{1/4}}\, \delta^3\, (\zeta^3 - 3\zeta) + o(\delta^3) \end{cases}$$

i.e. the blown up curve is

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2 \\ \tilde{y}(\zeta) = \zeta^3 - 3\zeta. \end{cases}$$

Not surprisingly, we recognize the polynomial $Q_1(\zeta) = \zeta^3 - 3\zeta$ of eq.(V-2-3).

Applying theorem 3.1, for example, we find for the first few n and g:

$$\tilde{\omega}_3^{(0)}(\zeta_1, \zeta_2, \zeta_3) = -\frac{1}{6}\, \frac{1}{\zeta_1^2 \zeta_2^2 \zeta_3^2} \tag{V-3-6}$$

$$\tilde{\omega}_1^{(1)}(\zeta) = -\frac{1}{(12)^2}\, \frac{\zeta^2 + 3}{\zeta^4} \tag{V-3-7}$$

$$\tilde{\omega}_2^{(1)}(\zeta_1, \zeta_2) = \frac{15\zeta_1^4 + 15\zeta_2^4 + 9\zeta_1^2\zeta_2^2 + 6\zeta_1^4\zeta_2^2 + 6\zeta_1^2\zeta_2^4 + 2\zeta_1^4\zeta_2^4}{2^5\, 3^3\, \zeta_1^6\, \zeta_2^6} \tag{V-3-8}$$

$$\tilde{\omega}_1^{(2)}(\zeta) = -7\, \frac{135 + 87\zeta^2 + 36\zeta^4 + 12\zeta^6 + 4\zeta^8}{2^{10}\, 3^5\, \zeta^{10}} \tag{V-3-9}$$

$$\tilde{\omega}_4^{(0)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \frac{1}{9\, \zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2} \left(1 + 3\sum_i \frac{1}{\zeta_i^2}\right)$$

$$\tilde{\omega}_5^{(0)}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5) = \frac{1}{9\;\zeta_1^2\zeta_2^2\zeta_3^2\zeta_4^2\zeta_5^2}\left(1+3\sum_i \frac{1}{\zeta_i^2}+6\sum_{i<j}\frac{1}{\zeta_i^2\zeta_j^2}+5\sum_i\frac{1}{\zeta_i^4}\right)$$

etc...

Using theorem III.4.7, we have

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z\to\pm 1}\omega_1^{(g)}(z)\,dz\;\;\ln z.$$

To leading order in δ, only the residue at $z=+1$ contributes, and writing $\ln z = \ln(1+\delta\zeta) = \delta\zeta + O(\delta)^2$, we get

$$\frac{\partial F_g}{\partial t} \sim -\,t^{1-2g}\,\delta^{5(1-2g)+1}\operatorname*{Res}_{\zeta\to 0}\tilde{\omega}_1^{(g)}(\zeta)\,\zeta\,d\zeta.$$

Similarly, taking a second derivative gives

$$\frac{\partial^2 F_g}{\partial t^2} \sim t^{-2g}\,\delta^{2-10g}\operatorname*{Res}_{\zeta_1\to 0}\operatorname*{Res}_{\zeta_2\to 0}\tilde{\omega}_2^{(g)}(\zeta_1,\zeta_2)\,\zeta_1\,d\zeta_1\,\zeta_2\,d\zeta_2.$$

and a third derivative

$$\frac{\partial^3 F_g}{\partial t^3} \sim -t^{-1-2g}\,\delta^{3-5(1+2g)}\operatorname*{Res}_{\zeta_1\to 0}\operatorname*{Res}_{\zeta_2\to 0}\operatorname*{Res}_{\zeta_3\to 0}\tilde{\omega}_3^{(g)}(\zeta_1,\zeta_2,\zeta_3)\,\zeta_1\,d\zeta_1\,\zeta_2\,d\zeta_2\,\zeta_3\,d\zeta_3.$$

From eq.V-3-6, we thus get

$$\frac{\partial^3 F_0}{\partial t^3} \sim -\frac{\delta^{-2}}{6t} \quad\longrightarrow\quad \frac{\partial^2 F_0}{\partial t^2} \sim -\,\frac{\delta^2}{2},$$

and using eq.V-3-8:

$$\frac{\partial^2 F_1}{\partial t^2} \sim \frac{\delta^{-8}}{2^4\,3^3\,t^2}$$

as well as using eq.V-3-9:

$$\frac{\partial F_2}{\partial t} \sim \frac{7\;\delta^{-14}}{2^8\;3^5\;t^3} \quad\longrightarrow\quad \frac{\partial^2 F_2}{\partial t^2} \sim \frac{49\;\delta^{-18}}{2^8\,3^6\,t^4}.$$

We define u_g such that

$$\frac{\partial^2 F_g}{\partial t^2} \sim u_g\;\frac{\delta^{2-10g}}{t^{2g}}$$

i.e.

$$u_0 = -\,\frac{1}{2}\,,\quad u_1 = \frac{1}{2^4\;3^3}\,,\quad u_2 = \frac{49}{2^8\;3^6}\,,\;\ldots$$

We may thus verify that the second derivative of the free energy:

$$u(s) = \sum_{g=0}^{\infty} s^{(1-5g)/2}\,u_g$$

satisfies the Painlevé I equation to the first orders:

$$2u^2 + \frac{1}{3^3}u'' = \frac{1}{2}s + o(s^{-4}).$$

Our goal now, is to prove that $u(s)$ satisfies Painlevé I to all orders.

4 Minimal models

The goal of this section is to prove that the following formal series

$$\ln \tau = \sum_g N^{2-2g}\,\tilde{F}_g$$

whose coefficients $\tilde{F}_g$ are the generating functions of large maps, is a formal **Tau-function** for the m^{th} reduction of the Kordeweg-De-Vries (**KdV**) hierarchy of integrable equations. That reduction of KdV is also called the $(2m+1, 2)$ minimal model in the context of conformal field theory. It can be obtained from Liouville conformal field theory coupled to 2D gravity.

In some sense, we obtain an argument towards the idea that large maps should be related to Liouville gravity.

4.1 Introduction to Minimal models

There exists several equivalent definitions of minimal models coupled to gravity. Here we shall adopt the approach of Douglas and Shenker in 1990 [28]. Minimal models correspond to representations of the conformal group in 2 dimensions. They are classified by two integers (p, q), and their central charge is:

$$c = 1 - 6\frac{(p-q)^2}{pq}.$$

Some of them have received special names (see [42]):

- $(1, 2)$ = Airy, $c = -2$ (related to Tracy-Widom law [80])
- $(3, 2)$ = pure gravity, $c = 0$
- $(5, 2)$ = Lee-Yang edge singularity, $c = -\frac{22}{5}$
- $(4, 3)$ = Ising, $c = \frac{1}{2}$
- $(6, 5)$ = Potts-3, $c = \frac{4}{5}$

Minimal models can also be viewed as finite reductions of the Kadamtsev-Petviashvili (KP) integrable hierarchy of partial differential equations [8, 54].

The case $q = 2$ is a little bit simpler to address, and is a reduction of the Korteweg de Vries (KdV) hierarchy [8, 48, 59].

The KdV hierarchy, and the minimal models $(p, 2)$ have generated a huge amount of works, and have been presented in many different (but equivalent) formulations. For instance in terms of a string equation for differential operators, in terms of a Lax pair, in terms of commuting hamiltonians, in terms of Schrödinger equation, in terms of Hirota equations, in terms of isomonodromic systems, in terms of Riemann Hilbert problems, in terms of tau functions, in terms of Grasman manifolds, in terms of Yang-Baxter equations, ...etc, see [8] for a comprehensive lecture.

All those formulations are equivalent, and let us recall some of the well known features of the $(p, 2)$ reduction of KdV (see [8, 27]), presented in a way convenient for our purposes.

4.2 String equation

The KdV minimal model $(p, 2)$ with $p = 2m + 1$, coupled to gravity, was formulated in terms of a "string equation" by Douglas and Shenker in 1990 [28]. Let P, Q two differential operators of respective orders p and 2, satisfying the so-called "string equation":

$$[P, Q] = \frac{1}{N}\,\mathrm{Id} \tag{V-4-1}$$

$$Q = d^2 - 2u(s) \qquad , \qquad P = d^p - p\,u\,d^{p-2} + \dots \qquad , \qquad d = \frac{1}{N}\,\frac{d}{ds}$$

$\frac{1}{N}$ is a redundant parameter, which can be absorbed by a redefinition of s and u, but we prefer to keep it to play the role of a scaling parameter which can be sent to zero to get the "classical limit".

In all this chapter, we shall denote with a dot the derivative with respect to s: $df/ds = \dot{f}$ in order to shorten notations. The prime will be reserved to derivatives with respect to the spectral parameter $df/dx = f'$.

Solution of the string equation

The general solution of the string equation eq.(V-4-1) is known. Let us describe it below.

Definition 4.1 *Let $(Q^{j+1/2})_+$ be the unique differential operator of order $2j+1$, such that:*

$$\mathrm{order}[((Q^{j+1/2})_+)^2 - Q^{2j+1}] \leq 2j.$$

For example:

$$(Q^{1/2})_+ = d \qquad , \qquad (Q^{3/2})_+ = d^3 - 3ud - \frac{3\dot{u}}{2N},$$

$$(Q^{5/2})_+ = d^5 - 5ud^3 + \frac{15}{2}\,u^2 d - \frac{15\dot{u}}{2N}\,d^2 - \frac{25\ddot{u}}{4N^2}\,d - \frac{15}{8N^3}\,\dddot{u} + \frac{15u\dot{u}}{2N} \qquad , \qquad \dots$$

Lemma 4.1 *It is a classical result (see [42]) that it satisfies:*

$$[(Q^{j-1/2})_+, Q] = \frac{1}{N}\,\frac{d}{dt}(R_j(u(s))) \tag{V-4-2}$$

where the right hand side is a function (a differential operator of order 0).

proof:

We propose the proof of this lemma as an exercise at the end of this chapter, and we give some hints of how to do it. □

The coefficients $R_j(u)$ are called the Gelfand-Dikii differential polynomials [42]. They can be obtained by a recursion.

Definition 4.2 (Gelfand-Dikii polynomials) *The Gelfand-Dikii differential polynomials are defined by the recursion:*

$$R_0 = 2 \qquad , \qquad \dot{R}_{j+1} = -2u\dot{R}_j - \dot{u}R_j + \frac{1}{4\,N^2}\dddot{R}_j. \tag{V-4-3}$$

and by the condition that R_j is homogenous of degree j in u with the grading convention that $\dot{} = \partial/\partial s$ has the same grading as $\sqrt{u}$.

The first few of them are:

$$\begin{aligned}
&R_0 = 2 \\
&R_1 = -2u \\
&R_2 = 3\,u^2 - \tfrac{1}{2\,N^2}\ddot{u} \\
&R_3 = -5u^3 + \frac{5}{2N^2}u\ddot{u} + \frac{5}{4N^2}\dot{u}^2 - \frac{1}{8N^4}\ddddot{u} \\
&\vdots
\end{aligned}$$

and in general:

$$\begin{aligned}
R_j(u) \quad = \quad & \frac{2\,(-1)^j\,(2j-1)!!}{j!}\left[u^j - \frac{j(j-1)}{12N^2}\,u^{j-2}\ddot{u}\right. \\
& \left.-\frac{j(j-1)(j-2)}{24N^2}\,u^{j-3}\dot{u}^2\right] + \quad \ldots \quad - \frac{2}{(2N)^{2j-2}}\,u^{(2j-2)}.
\end{aligned} \tag{V-4-4}$$

Lemma 4.2 *Any solution of the string equation*

$$[P, Q] = \frac{1}{N}\,\mathrm{Id}$$

where $Q = d^2 - 2u$ and $P = d^{2m+1} + \ldots$, can be written:

$$P = \sum_{j=0}^{m} \tilde{t}_j (Q^{j+1/2})_+ + \sum_{j=0}^{m-1} c_j Q^j \qquad , \qquad \tilde{t}_m = 1$$

where c_j, $\tilde{t}_j$ are constants (independent of s) and $u(s)$ is a solution of the non-linear differential equation:

$$\boxed{\sum_{j=0}^{m} \tilde{t}_j R_{j+1}(u) = s.} \tag{V-4-5}$$

This equation has the Painlevé property

proof:
The proof that the solution takes that form is obvious from lemma 4.1. The fact that the equation satisfies the Painlevé property is beyond the scope of this book, and we shall not use it here. We refer the reader to [23] for more details about the Painlevé property. □

The coefficients c_j associated to Q^j will play no role in what follows, because $[Q^j, Q] = 0$, so from now on, we shall choose $c_j = 0$.

Remark 4.1 Since $R_0 = 2$, we see that we can identify s with $s = -2\tilde{t}_{-1}$.

Examples:
• For Airy $p = 1$, the equation for u is:

$$-2u = s. \tag{V-4-6}$$

• For pure gravity $p = 3$, this is the Painlevé I equation:

$$3\,u^2 - \frac{1}{2N^2}\ddot{u} - 2\tilde{t}_0 u = s. \tag{V-4-7}$$

• For Lee-Yang $p = 5$, we have:

$$-5u^3 + \frac{5}{2N^2}u\ddot{u} - \frac{5}{4N^2}\dot{u}^2 - \frac{1}{8N^4}\ddddot{u} + \tilde{t}_1(3\,u^2 - \frac{1}{2N^2}\ddot{u}) - 2\tilde{t}_0 u = s. \tag{V-4-8}$$

4.3 Lax pair

Consider the following matrices:

Definition 4.3

$$\mathcal{R}(x,s) = \begin{pmatrix} 0 & 1 \\ x + 2u(s) & 0 \end{pmatrix},$$

and for any integer k:

$$\mathcal{D}_k(x,s) = \begin{pmatrix} A_k & B_k \\ C_k & -A_k \end{pmatrix},$$

where $A_k(x,s), B_k(x,s), C_k(x,s)$ are polynomials of respective degree $k-1, k, k+1$ in x, which are defined by ($R_j(u)$ is the j^{th} Gelfand-Dikii polynomial, cf def 4.2):

$$B_k(x,s) = \frac{1}{2}\sum_{j=0}^{k} x^{k-j}\; R_j(u) \qquad , \qquad A_k = -\frac{1}{2N}\dot{B}_k \qquad , \qquad C_k = (x+2u)\,B_k + \frac{1}{N}\dot{A}_k.$$

The recursion relation eq.(V-4-3) implies that B_k satisfies the equation:

$$2\dot{u}B_k + 2(x+2u)\dot{B}_k - \frac{1}{2N^2}\dddot{B}_k = -\dot{R}_{k+1}(u)$$

and we see that

Lemma 4.3 *the matrix $\mathcal{D}_k(x,t)$ satisfies:*

$$\frac{1}{N}\frac{\partial}{\partial s}\mathcal{D}_k(x,s) + [\mathcal{D}_k(x,s), \mathcal{R}(x,s)] = -\frac{1}{N}\dot{R}_{k+1}(u)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{V-4-9}$$

the right hand side is independent of x, and is proportional to $\frac{\partial}{\partial x}\mathcal{R}(x,s)$.

This equation is called a "Lax equation".

4.4 Lax equation

Therefore we have obtained that, if u is a solution of the string equation eq.(V-4-5), then, the matrix:

$$\mathcal{D}(x,s) = \sum_{j=0}^{m} \tilde{t}_j \mathcal{D}_j(x,s) \qquad , \qquad \tilde{t}_m = 1$$

satisfies the Lax equation:

Proposition 4.1 *The matrices $\mathcal{D}(x,s)$ and $\mathcal{R}(x,s)$ form a Lax pair, they satisfy the Lax equation*

$$\frac{1}{N}\frac{\partial}{\partial s}\mathcal{D}(x,s) + [\mathcal{D}(x,s), \mathcal{R}(x,s)] = -\frac{1}{N}\frac{\partial}{\partial x}\mathcal{R}(x,s) \tag{V-4-10}$$

which can also be written as

$$\left[\frac{1}{N}\frac{\partial}{\partial x} + \mathcal{D}(x,s), \mathcal{R}(x,s) - \frac{1}{N}\frac{\partial}{\partial s}\right] = 0 \tag{V-4-11}$$

This relation means that the operator $\frac{1}{N}\frac{\partial}{\partial x} + \mathcal{D}(x,s)$ is a Lax operator [8].

4.5 The linear ψ system

The Lax equation eq.(V-4-11) is the compatibility condition, which says that the following two differential systems have a common solution $\Psi(x,s)$:

$$\frac{1}{N}\frac{d}{dx}\Psi(x,s) = -\mathcal{D}(x,s)\Psi(x,s) \qquad , \qquad \frac{1}{N}\frac{d}{ds}\Psi(x,s) = \mathcal{R}(x,s)\Psi(x,s) \tag{V-4-12}$$

and $\Psi(x,s)$ is a matrix such that:

$$\Psi(x,s) = \begin{pmatrix} \psi & \phi \\ \tilde{\psi} & \tilde{\phi} \end{pmatrix} \qquad , \qquad \det \Psi = 1. \tag{V-4-13}$$

In particular this implies the Schrödinger equation for ψ:

$$\frac{1}{N^2}\ddot{\psi}(x,s) = (x + 2u(s))\,\psi(x,s) \tag{V-4-14}$$

where s can be interpreted as the space variable, $u(s)$ is the potential, and x the energy. This is why x is often called the "**spectral parameter**". $\hbar = 1/N$ can be interpreted as the Planck constant and this is why the limit $N \to \infty$ is called the "**classical limit**".

It is possible to normalize $\det\Psi = 1$, because $\frac{1}{N}d/ds \ln\det\Psi = \operatorname{tr}\mathcal{R}(x,s) = 0$, and thus $\det\Psi(x,s)$ is independent of s, similarly, $\frac{1}{N}d/dx \ln\det\Psi = -\operatorname{tr}\mathcal{D}(x,s) = 0$, and thus $\det\Psi(x,s)$ is independent of x, i.e. it is a constant, and up to a choice of normalization, it can be chosen equal to 1.

4.6 Kernel and correlators

Define

Definition 4.4 *the (generalized) Christoffel-Darboux kernel associated to the system $\mathcal{D}(x,s)$ is defined as*

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2) - \tilde{\psi}(x_1)\phi(x_2)}{x_1-x_2} = \frac{1}{x_1-x_2}\,\left(\Psi(x_1)^{-1}\,\Psi(x_2)\right)_{2,2}. \tag{V-4-15}$$

Remark 4.2 In fact, the actual Christoffel-Darboux kernel usually considered in the literature, is the $\left(\Psi(x_1)^{-1}\,\Psi(x_2)\right)_{2,1}$. It turns out that the 2 are related, and this one is more convenient for our purposes.

Definition 4.5 *We define the "***connected correlators***" by the "determinantal formulae":*

$$\hat{W}_1(x) = \lim_{x'\to x} K(x,x') - \frac{1}{x-x'} = \psi'(x)\tilde{\phi}(x) - \tilde{\psi}'(x)\phi(x) \tag{V-4-16}$$

and for $n \geq 2$:

$$\hat{W}_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} - (-1)^n \sum_{\sigma=\text{cyles}} \prod_{i=1}^{n} K(x_i, x_{\sigma(i)}) \tag{V-4-17}$$

where we take the sum over all cyclic permutations (i.e. σ has only one cycle).

For example:

$$\hat{W}_2(x_1,x_2) = -K(x_1,x_2)K(x_2,x_1) - \frac{1}{(x_1-x_2)^2},$$

$$\hat{W}_3(x_1,x_2,x_3) = K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1).$$

Although we have not written it explicitly, the kernel K and the correlators $\hat{W}_n$ depend on s.

Remark 4.3 Our goal in this section will be to prove that the correlators $\hat{W}_n$ defined from the minimal model, coincide with the correlators $\tilde{W}_n$ of section 3 defined from the double scaling limit of generating functions of large maps:

$$\hat{W}_n \overset{?}{=} \tilde{W}_n.$$

Definition 4.6 *The* non-connected *correlators are defined by:*

$$\hat{W}_n(x_1,\dots,x_n)_{n.c.} = \sum_{\mu\vdash\{x_1,\dots,x_n\}} \prod_{i=1}^{\ell(\mu)} \hat{W}_{|\mu_i|}(\mu_i),$$

where the sum runs over all partitions $\mu = (\mu_1,\dots,\mu_{\ell(\mu)})$ of $\{x_1,\dots,x_n\}$ into non-empty disjoint subsets. In other words, the connected $\hat{W}_n$'s are the cumulants of the non-connected ones.

For instance:

$$\hat{W}_2(x_1,x_2)_{n.c.} = \hat{W}_2(x_1,x_2) + \hat{W}_1(x_1)\hat{W}_1(x_2),$$

$$\begin{aligned}\hat{W}_3(x_1,x_2,x_3)_{n.c.} &= \hat{W}_3(x_1,x_2,x_3) + \hat{W}_1(x_1)\hat{W}_2(x_2,x_3) + \hat{W}_1(x_2)\hat{W}_2(x_1,x_3) \\ &\quad + \hat{W}_1(x_3)\hat{W}_2(x_1,x_2) + \hat{W}_1(x_1)\hat{W}_1(x_2)\tilde{W}_1(x_3). \qquad \text{(V-4-18)}\end{aligned}$$

The formula eq.(V-4-17) is called "determinantal formula", because for the non-connected correlators, the sum over cyclic permutations in eq.(V-4-17) is replaced by a sum over all permutations, with their signature:

$$\hat{W}_n(x_1,\dots,x_n)_{n.c.} = \det{}'(K(x_i,x_j)) = \sum_\sigma{}' (-1)^\sigma \prod_i K(x_i, x_{\sigma(i)})$$

where $\det'$ and $\sum'$ signify that whenever the permutation σ has a fixed point if $\sigma(i) = i$ we must replace the ill-defined $K(x_i,x_i)$ by $\hat{W}_1(x_i)$, and whenever the permutation σ has a cycle of length 2, i.e. $\sigma(i) = j$ and $\sigma(j) = i$, we replace $K(x_i,x_j)K(x_j,x_i)$ by $-\hat{W}_2(x_i,x_j)$, see [10].

For instance $\hat{W}_{3,\,n.c.}$ is the sum of 6 terms coming from the 6 permutations:

$$\begin{aligned}\hat{W}_{3,\,n.c.}(x_1,x_2,x_3) &= \det{}' \begin{pmatrix} K(x_1,x_1) & K(x_1,x_2) & K(x_1,x_3) \\ K(x_2,x_1) & K(x_2,x_2) & K(x_2,x_3) \\ K(x_3,x_1) & K(x_3,x_2) & K(x_3,x_3)\end{pmatrix} \\ &= \hat{W}_1(x_1)\hat{W}_1(x_2)\hat{W}_1(x_3) + \hat{W}_1(x_1)\hat{W}_2(x_2,x_3) + \hat{W}_1(x_2)\hat{W}_2(x_1,x_3) \\ &\quad + \hat{W}_1(x_3)\hat{W}_2(x_1,x_2) + K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) \\ &\quad + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1) \qquad \text{(V-4-19)}\end{aligned}$$

which coincides with eq.(V-4-18).

Alternative definition of the correlators

Notice that:

$$K(x,x') = \frac{(\psi(x)\tilde{\phi}(x') - \tilde{\psi}(x)\phi(x'))}{x-x'} = \frac{1}{x-x'}\,\left(\Psi(x)^{-1}\,\Psi(x')\right)_{2,2}$$

and thus

$$K(x,x')K(x',x'') = \frac{1}{(x-x')(x'-x'')}\,\left(\Psi(x)^{-1}\,\Psi(x')E\,\Psi(x')^{-1}\,\Psi(x'')\right)_{2,2}$$

where E is a matrix which projects on the $(2,2)$ coefficient:

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This leads to define

Definition 4.7 *The projector $M(x)$:*

$$M(x) = \Psi(x)\, E\, \Psi(x)^{-1} = \begin{pmatrix} -\tilde{\psi}(x)\phi(x) & \psi(x)\phi(x) \\ -\tilde{\psi}(x)\tilde{\phi}(x) & \psi(x)\tilde{\phi}(x) \end{pmatrix}$$

The matrix $M(x)$ is a projector, it satisfies

$$M(x)^2 = M(x) \qquad , \qquad \mathrm{Tr}\, M(x) = 1 \qquad , \qquad \det M(x) = 0.$$

Thanks to that matrix $M(x)$, we can rewrite any cyclic product of $K(x_i, x_{\sigma(i)})$ as a cyclic product of matrices $M(x)$:

$$\prod_i K(x_i, x_{\sigma(i)}) = \frac{\mathrm{Tr}\, \prod_i M(x_{\sigma^i(1)})}{\prod_i (x_i - x_{\sigma(i)})}$$

For example:

$$\hat{W}_2(x, x') = -\, K(x, x')K(x', x) - \frac{1}{(x - x')^2} = \frac{\mathrm{Tr}\, M(x)M(x')}{(x - x')^2} - \frac{1}{(x - x')^2}$$

and

$$K(x, x')K(x', x'')K(x'', x) = \frac{\mathrm{Tr}\, M(x)M(x')M(x'')}{(x - x')(x' - x'')(x'' - x)}$$

It follows that

$$\hat{W}_3(x, x', x'') = \frac{\mathrm{Tr}\, \left(M(x)M(x')M(x'') - M(x)M(x'')M(x')\right)}{(x - x')(x' - x'')(x'' - x)} = \frac{\mathrm{Tr}\, M(x)\, [M(x'), M(x'')]}{(x - x')(x' - x'')(x'' - x)}.$$

And in general the correlators are:

Theorem 4.1

$$\hat{W}_1(x) = N\, \mathrm{Tr}\, \mathcal{D}(x)\, M(x)$$

$$\hat{W}_2(x_1, x_2) = \frac{\mathrm{Tr}\, M(x_1)M(x_2)}{(x_1 - x_2)^2} - \frac{1}{(x_1 - x_2)^2}$$

and for $n \geq 3$

$$\hat{W}_n(x_1, \dots, x_n) = (-1)^{n-1} \sum_{\sigma = \text{cyclic}} \frac{\mathrm{Tr}\, \prod_{i=0}^{n-1} M(x_{\sigma^i(1)})}{\prod_{i=1}^{n} (x_i - x_{\sigma(i)})}$$

Loop insertion

We shall define a "loop insertion operator" δ_x acting as a derivation (i.e. satisfying Leibniz's chain rule) on the functions $\psi, \tilde{\psi}, \phi, \tilde{\phi}$, and inserting functions of more variables (whence the name insertion).

In that purpose we first need to define formally derivative acting on a set of functions, without evaluating the derivatives. This is the notion of the Picard-Vessiot ring.

let $\mathbb{F}_n = \mathbb{C}(x_1, \dots, x_n)$ be the field of rational functions of n variables, and $\mathbb{F}_\infty$ the projective limit $n \to \infty$.

Definition 4.8 *Let $\mathcal{A}$ be the Picard-Vessiot differential ring over $\mathbb{F}$ freely generated by the symbols $\psi(x), \tilde{\psi}(x), \phi(x), \tilde{\phi}(x)$, and quotiented by the relation $\psi(x)\tilde{\phi}(x) - \tilde{\psi}(x)\phi(x) = 1$.*

This means that $\mathcal{A}$ is generated by all the symbols $\psi(x), \tilde{\psi}(x), \phi(x), \tilde{\phi}(x)$, and their derivatives with respect to any $\partial/\partial x_i$ and $\partial/\partial s$, and is the set of all their sums and products.

We also define its n-dimensional analogue, $\mathcal{A}_n$ to be the Picard-Vessiot differential ring with n variables.
It is the differential ring over $\mathbb{F}_n$ freely generated by the symbols $\psi(x_i), \tilde{\psi}(x_i), \phi(x_i), \tilde{\phi}(x_i)$, $i = 1, \dots, n$, and quotiented by the n relations $\psi(x_i)\tilde{\phi}(x_i) - \tilde{\psi}(x_i)\phi(x_i) = 1$.
Let $\mathcal{A}_\infty$ its $n \to \infty$ projective limit.

Then, we define a loop insertion operator, as an operator $\delta : \mathcal{A}_n \to \mathcal{A}_{n+1}$, by:

Definition 4.9 *Let $U(x_1) \in M_2(\mathcal{A}_1)$ an arbitrary 2×2 matrix whose elements belong to the Picard-Vessiot ring $\mathcal{A}_1$, i.e. only function of one variable x_1.*
We say that δ, acting in $\mathcal{A}_\infty$ is a "loop insertion operator" if it satisfies:

- *δ sends $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$.*
- *δ annihilates $\mathbb{F}_\infty$, i.e. $\mathbb{F}_\infty \subset \operatorname{Ker}\delta$.*
- *δ is a derivation, it satisfies the Leibniz rule $\delta_x(fg) = f\delta_x g + g\delta_x f$.*
- *its action on the generators of $\mathcal{A}$ is*

$$\delta_{x'}\Psi(x) = \frac{M(x')}{x - x'}\,\Psi(x) + U(x')\,\Psi(x)$$

- *it commutes with the derivations d/dx_i and d/ds:*

$$\left[\delta_{x_j}, \frac{d}{dx_i}\right] = 0 \qquad , \qquad \left[\delta_{x_j}, \frac{d}{ds}\right] = 0$$

  This is equivalent to requiring

$$\delta_{x'}\mathcal{D}(x) = [\frac{M(x')}{x - x'} + U(x'), \mathcal{D}(x)] + \frac{1}{N}\,\frac{M(x')}{(x - x')^2}$$

$$\delta_{x'}\mathcal{R}(x) = [M(x'), \frac{\mathcal{R}(x) - \mathcal{R}(x')}{x - x'}] + [U(x'), \mathcal{R}(x)] + \frac{1}{N}\,\dot{U}(x')$$

- *The δ_{x_i}'s comute together:*

$$[\delta_{x_i}, \delta_{x_j}] = 0$$

  This last requirement is equivalent to demand that

$$\delta_x U(y) - \delta_y U(x) = [U(x), U(y)] + \frac{U(x) - U(y)}{y - x}.$$

(In order for this definition to be meaningful, one has to check that the action of δ is compatible with the relations $\psi(x_i)\tilde{\phi}(x_i) - \tilde{\psi}(x_i)\phi(x_i) = 1$, which we leave to the reader as an easy exercise).

The existence of an insertion operator is not automatic and not trivial.

However, in our case, such an operator exists:

Proposition 4.2 *The following choice for $U(x)$ fulfills all the requirements:*

$$U(x) = \begin{pmatrix} 0 & 0 \\ \psi(x)\phi(x) & 0 \end{pmatrix}$$

provided that we define

$$\delta_x\, u(s) = \psi(x)\tilde{\phi}(x) + \tilde{\psi}(x)\phi(x) = \frac{1}{N}\frac{d}{ds}\,\psi(x)\phi(x). \tag{V-4-20}$$

proof:

We leave it as an exercise. □

The main properties of the insertion operator are

Proposition 4.3 *The kernel K is self–reproducing:*

$$\delta_{x'} K(x, x'') = -K(x, x')K(x', x'')$$

This implies that

$$\delta_{x_{n+1}} W_n(x_1, \ldots, x_n) = W_{n+1}(x_1, \ldots, x_n, x_{n+1}) + \frac{\delta_{n,1}}{(x_1 - x_2)^2}$$

We also have that:

$$\delta_{x'} M(x) = \frac{[M(x'), M(x)]}{x - x'} + [U(x'), M(x)]$$

$$\delta_{x'} \mathcal{D}(x) = \frac{[M(x'), \mathcal{D}(x)]}{x - x'} + [U(x'), \mathcal{D}(x)] + \frac{1}{N}\,\frac{M(x')}{(x - x')^2} \tag{V-4-21}$$

proof:

Those relations are easy to derive from the definition of δ, we leave it as an exercise for the reader. □

Loop equations

Theorem 4.2 (Loop equations) *(proved in [10]):*

the quantity

$$\begin{aligned} & P_n(x; x_1, \ldots, x_n) \\ = \quad & \hat{W}_{n+2,\, n.c.}(x, x, x_1, \ldots, x_n) \\ & + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\, \frac{\hat{W}_n(x, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - \hat{W}_n(x_1, \ldots, x_n)}{x - x_j} \end{aligned}$$

$(V - 4 - 22)$

is a polynomial of the variable x.

This assertion is highly non trivial because none of the terms in the right hand side are polynomials of x, they involve functions $\psi(x)$ for instance. Only this very combination is polynomial.
proof:

The full proof can be found in [10, 11]. Let us give a hint of the proof.

The case $n=0$ is very easy, one can explicitly compute:

$$P_0(x) = \hat{W}_2(x,x) + \hat{W}_1(x)^2 = -N^2\,\det\mathcal{D}(x,s) = \frac{N^2}{2}\,\mathrm{Tr}\,\mathcal{D}(x,s)^2$$

which is indeed a polynomial of x.

The cases $n \geq 1$ can be obtained from $n=0$ by recursively applying δ_{x_i}. Indeed, we have:

$$P_{n+1}(x;x_1,\dots,x_{n+1}) = \delta_{x_{n+1}}\,P_n(x;x_1,\dots,x_n) - \frac{\partial}{\partial x_{n+1}}\,\frac{\hat{W}_n(x_{n+1},x_1,\dots,x_n)}{x-x_{n+1}}$$

Then observe from eq.(V-4-21) that $\delta_{x_{n+1}}\,\mathcal{D}(x)$ is a rational fraction of x, containing only elements of $\mathcal{A}_n$, and P_n can have no other pole than $x=\infty$.

$\square$

For example, we have that

$$\begin{aligned}
P_1(x;x_1) &= \delta_{x_1}P_0(x) - \frac{\partial}{\partial x_1}\,\frac{\hat{W}_1(x_1)}{x-x_1} \\
&= \frac{N^2}{2}\,\delta_{x_1}\left(\mathrm{Tr}\,\mathcal{D}(x,s)^2\right) - N\frac{\partial}{\partial x_1}\,\frac{1}{x-x_1}\;\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1) \\
&= N^2\;\mathrm{Tr}\,\mathcal{D}(x,s)\,\delta_{x_1}\mathcal{D}(x,s) - N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1)}{(x-x_1)^2} \\
&\quad -N\,\frac{\mathrm{Tr}\,\mathcal{D}'(x_1,s)\,M(x_1)}{x-x_1} - N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M'(x_1)}{x-x_1} \\
&= N^2\;\mathrm{Tr}\,\mathcal{D}(x,s)\,\left(\frac{[M(x_1),\mathcal{D}(x,s)]}{x-x_1} + [U(x_1),\mathcal{D}(x,s)] + \frac{1}{N}\,\frac{M(x_1)}{(x-x_1)^2}\right) \\
&\quad -N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1)}{(x-x_1)^2} - N\,\frac{\mathrm{Tr}\,\mathcal{D}'(x_1,s)\,M(x_1)}{x-x_1} \\
&\quad +N^2\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,[\mathcal{D}(x_1,s),M(x_1)]}{x-x_1} \\
&= N\;\mathrm{Tr}\,\mathcal{D}(x,s)\,\frac{M(x_1)}{(x-x_1)^2} - N\,\frac{\mathrm{Tr}\,\mathcal{D}(x_1,s)\,M(x_1)}{(x-x_1)^2} - N\,\frac{\mathrm{Tr}\,\mathcal{D}'(x_1,s)\,M(x_1)}{x-x_1}
\end{aligned}$$

and one observes that $(\mathcal{D}(x,s) - \mathcal{D}(x_1,s) - (x-x_1)\mathcal{D}'(x_1,s))/(x-x_1)^2$ is indeed a polynomial of x.

4.7 Example: (1,2) minimal model, the Airy kernel

Let us write the $(1,2)$ model, i.e. $m=0$. We have:

$$P = d \qquad , \qquad Q = d^2 - 2u$$

the string equation is:

$$[P, Q] = -\frac{2}{N}\,\dot{u} = \frac{1}{N}$$

i.e.

$$u(s) = -\frac{s}{2} = \tilde{t}_{-1}$$

The Lax pair is:

$$\mathcal{D}(x, s) = \begin{pmatrix} 0 & 1 \\ x - s & 0 \end{pmatrix} \quad , \quad R(x, s) = \begin{pmatrix} 0 & 1 \\ x - s & 0 \end{pmatrix}$$

The differential system is:

$$\frac{1}{N}\frac{d}{dx}\,\Psi(x, s) = -\begin{pmatrix} 0 & 1 \\ x - s & 0 \end{pmatrix}\Psi(x, s)$$

i.e.

$$\psi'' = N^2(x - s)\psi$$

whose solution is the Airy function (The Airy function is solution of $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$, see textbooks on classical functions [1]) rescaled by $N^{2/3}$:

$$\psi(x, s) = \mathrm{Ai}(N^{\frac{2}{3}}(x - s)) \quad , \quad \tilde{\psi}(x, s) = -N^{-1/3}\,\mathrm{Ai}'(N^{\frac{2}{3}}(x - s))$$

and the other independent solution is the "BAiry" function [1]:

$$\phi(x, s) = -\pi\,N^{1/3}\mathrm{Bi}(N^{\frac{2}{3}}(x - s)) \quad , \quad \tilde{\phi}(x, s) = \pi\,\mathrm{Bi}'(N^{\frac{2}{3}}(x - s))$$

where in the litterature, Bi is normalized so that $\mathrm{Ai}\,\mathrm{Bi}' - \mathrm{Ai}'\,\mathrm{Bi} = 1/\pi$.

The Christoffel–Darboux kernel is thus the famous Airy kernel [80]:

$$K_{\mathrm{Airy}}(s + N^{-2/3}x_1, s + N^{-2/3}x_2) = \pi\,\frac{\mathrm{Ai}(x_1)\mathrm{Bi}'(x_2) - \mathrm{Ai}'(x_1)\mathrm{Bi}(x_2)}{x_1 - x_2}$$

and this is why the $(1, 2)$ minimal model coupled to gravity, is sometimes called the "Airy model". In fact, since $\mathrm{Ai}(x) \propto \mathrm{Bi}(x\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}) - \mathrm{Bi}(x\,\mathrm{e}^{\frac{-2\pi\mathrm{i}}{3}})$, by taking a linear combination of rotations we can replace Bi by Ai. This is the usual convention for defining the Airy kernel, ours is slightly more general and recovers the standard one by taking the difference after rotations by angles $\pm 2\pi/3$.

Remark 4.4 The Airy kernel plays a very important role in many problems, in particular in the universal laws of extreme values, related to the Tracy-Widom law [80]. We mention this, not as a coincidence, but because, as we have seen in chapter II counting maps is closely related to random matrices, and the asymptotic limit is closely related to the eigenvalue statistics at the end of the spectrum.

So it is very natural that large maps can be related to Tracy-Widom law of extreme eigenvalues.

Let us parametrize

$$\mathrm{Ai}(x) = \frac{\sqrt{f(x)}}{2\sqrt{\pi}}\,\mathrm{e}^{-\int_\infty^x \frac{dx'}{f(x')}} \qquad , \qquad \mathrm{Bi}(x) = \frac{\sqrt{f(x)}}{\sqrt{\pi}}\,\mathrm{e}^{\int_\infty^x \frac{dx'}{f(x')}}.$$

The Airy equation $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$, implies that $f(x)$ satisfies the differential equation

$$xf^2 + \frac{f'^2}{4} - 1 = \frac{1}{2}\,ff''$$

and taking the derivative again, and after dividing by f'

$$\frac{1}{2}\,f''' = 2x\,f' + f.$$

One easily finds from this linear equation and from the leading behavior $f \sim 1/\sqrt{x}$, that:

$$f(x) = \frac{1}{\sqrt{x}} + \sum_{k=1}^\infty \frac{(6k-1)!!}{2^{5k}\,3^k\,k!}\,x^{-3k-\frac{1}{2}}.$$

Then, from eq.(V-4-16) we compute the 1-point function

$$\begin{aligned}
\hat{W}_1(s+N^{-2/3}x) &= \pi\,N^{2/3}\,\left(\mathrm{Ai}'(x)\mathrm{Bi}'(x) - x\,\mathrm{Ai}(x)\mathrm{Bi}(x)\right) \\
&= N^{2/3}\,\frac{1}{2\,f(x)}\,\left(\frac{f'^2(x)}{4} - 1 - x\,f(x)^2\right) \\
&= N^{2/3}\,\left(\frac{1}{4}\,f''(x) - x\,f(x)\right)
\end{aligned}$$

Taking the derivative again implies $N^{-4/3}\,\hat{W}_1'(s+N^{-2/3}x) = -f(x)/2$, and therefore

$$\hat{W}_1(x) = -N\sqrt{x-s} + \sum_{k=1}^\infty \frac{(6k-3)!!}{2^{5k}\,3^k\,k!}\,(x-s)^{-3k+1/2}\,N^{1-2k}.$$

We may write it:

$$\hat{W}_1(x) = \sum_{g=0}^\infty N^{1-2g}\,\hat{W}_1^{(g)}(x)$$

with

$$\hat{W}_1^{(0)}(x) = -\sqrt{x-s} \qquad , \qquad \hat{W}_1^{(g)}(x) = \frac{(6g-3)!!}{2^{5g}\,3^g\,g!}\,(x-s)^{-3g+1/2}.$$

The τ function (defined in the next section) is simply:

$$\tau = \mathrm{e}^{-\frac{N^2 s^3}{12}}.$$

For the Airy system, the polynomial of theorem 4.2 is simply:

$$P_n(x) = (x-s)\,\delta_{n,0}.$$

4.8 Tau function

The notion of Isomonodromic Tau-function was defined for any Lax pair by Jimbo-Miwa [52, 53]. In this book we shall not study in details why the Tau-function is a useful notion, we just mention that indeed it encodes most of the properties of an integrable system, it is a very fundamental notion. We refer the reader to literature on integrable systems for learning more about Tau–functions and their utility, see for instance [8, 50, 60, 61].

Let us describe how it is defined in our case. In order to define the Tau-function, we need to consider the large x formal asymptotic expansion of $\Psi(x)$.

First we define

$$T(x) = \left(\int^x Y(x')\, dx' \right)_+$$

where $Y(x) = \sqrt{A^2(x) + B(x)C(x)}$ is (up to a sign) the eigenvalue of $\mathcal{D}(x)$, and $()_+$ means the strictly positive part of the Laurent series in $\sqrt{x}$ (and thus it is independent of a choice of integration constant). By an easy induction, one sees that the large x formal asymptotics of $\Psi(x)$ is of the form

$$\Psi(x) \sim \frac{1}{\sqrt{2}} \begin{pmatrix} x^{-1/4} & -x^{-1/4} \\ x^{1/4} & x^{1/4} \end{pmatrix} \tilde{\Psi}(x)\, \mathrm{e}^{-N\sigma_3 T(x)} \qquad , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and where $\tilde{\Psi}(x) = \mathrm{Id} + O(1/\sqrt{x})$ is an analytical function of $\sqrt{x}$ near ∞:

$$\tilde{\Psi}(x) = \mathrm{Id} + \frac{v}{\sqrt{x}}\, \sigma_3 + \frac{v^2}{2x}\, \mathrm{Id} + \frac{u}{2x} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O(x^{-3/2}) \qquad , \qquad \frac{1}{N}\, \dot{v} = u, \quad \text{(V-4-23)}$$

Miwa-Jimbo [52,53] define the Tau-function $\tau(s)$ and its log, the free energy function $\mathcal{F}(s) = \ln \tau(s)$ such that:

$$\frac{\partial \mathcal{F}}{\partial s} = -N \operatorname*{Res}_{x\to\infty} \operatorname{Tr} \left(\Psi(x)^{-1}\, \Psi'(x)\, \sigma_3 \right) \frac{\partial T(x)}{\partial s}\, dx$$

First notice that

$$\begin{aligned} \operatorname{Tr} \left(\Psi(x)^{-1}\, \Psi'(x)\, \sigma_3 \right) &= \operatorname{Tr} \begin{pmatrix} \psi'\tilde{\phi} + \phi'\tilde{\psi} & -\psi'\phi - \phi'\psi \\ \tilde{\psi}'\tilde{\phi} + \tilde{\phi}'\tilde{\psi} & -\psi\tilde{\phi}' - \phi\tilde{\psi}' \end{pmatrix} \\ &= \psi'\tilde{\phi} - \phi\tilde{\psi}' + \phi'\tilde{\psi} - \psi\tilde{\phi}' \\ &= 2\, \hat{W}_1(x) \end{aligned}$$

which is a Laurent series in $\sqrt{x}$.

Therefore the definition of the Tau function is equivalent to

$$\dot{\mathcal{F}} = -2N \operatorname*{Res}_{x\to\infty} \hat{W}_1(x)\, \dot{T}(x)\, dx$$

Then, write $Y(x) = \sqrt{-\det \mathcal{D}} = \sqrt{\frac{1}{2} \operatorname{Tr} \mathcal{D}^2}$, so that

$$2Y(x)\, \frac{\partial Y(x)}{\partial s} = \operatorname{Tr} \mathcal{D}\, \dot{\mathcal{D}} = \operatorname{Tr} \mathcal{D}\, (-\mathcal{R}' - N\, [\mathcal{D}, \mathcal{R}]) = -\operatorname{Tr} \mathcal{D}\, \mathcal{R}' = -B(x)$$

i.e.

$$\begin{aligned}
\frac{\partial Y(x)}{\partial s} &= -\frac{B(x)}{2\,Y(x)} = -\frac{B}{2\sqrt{B\,C+A^2}} \\
&= -\frac{1}{2\sqrt{(x+2u)-\frac{1}{2N^2}\frac{\ddot{B}}{B}+\frac{1}{4N^2}\frac{\dot{B}^2}{B^2}}} \\
&= -\frac{1}{2\sqrt{x+2u}}\,(1+O(1/x^2))
\end{aligned}$$

Then, since $T(x) = \left(\int^x Y(x')dx'\right)_+$, by integration we find

$$\frac{\partial T(x)}{\partial s} = -\sqrt{x}.$$

In our case this leads to

$$N^{-1}\,\partial\mathcal{F}/\partial s = 2\,\mathop{\mathrm{Res}}_{x\to\infty} \hat{W}_1(x)\,\sqrt{x}\,dx$$

where $\hat{W}_1(x) = \psi'(x)\tilde{\phi}(x) - \tilde{\psi}'(x)\phi(x)$. Taking another derivative with respect to s, and using the $\partial/\partial s$ equation satisfied by $\Psi(x,s)$, we get

$$\begin{aligned}
N^{-1}\,\partial\hat{W}_1(x)/\partial s &= \tilde{\psi}'(x)\tilde{\phi}(x) + \psi'(x)(x+2u(s))\phi(x) \\
&\quad -((x+2u(s))\psi(x))'\phi(x) - \tilde{\psi}'(x)\tilde{\phi}(x) \\
&= -\psi(x)\phi(x),
\end{aligned}$$

and therefore:

$$N^{-2}\,\partial^2\mathcal{F}/\partial s^2 = -2\,\mathop{\mathrm{Res}}_{x\to\infty} \psi(x)\phi(x)\sqrt{x}\,dx$$

From the asymptotic expansion eq.(V-4-23), one has

$$\psi(x)\phi(x) \sim -\frac{1}{2\sqrt{x}}\,(1-\frac{u}{x}+O(x^{-3/2}))$$

and thus

$$N^{-2}\,\partial^2\mathcal{F}/\partial s^2 = \mathop{\mathrm{Res}}_{x\to\infty}(1-\frac{u}{x}+O(x^{-3/2}))\,dx = u$$

And therefore we find

$$N^{-2}\,\partial^2\mathcal{F}/\partial s^2 = u(s).$$

Therefore we have just recovered the Its-Matveev's equation [50]:

Theorem 4.3 *The Tau-function of the integrable system defined by the Lax pair $(\mathcal{D},\mathcal{R})$, is such that $u(s)$ is the second derivative of $\ln\tau$:*

$$\tau(s) = \mathrm{e}^{N^2\,h(s)} \qquad , \qquad \frac{\partial^2\,h(s)}{\partial\,s^2} = u(s),$$

and $u(s)$ is solution of the Gelfan-Dikii equation eq.(V-4-5) of lemma 4.2:

$$\sum_{j=0}^{m}\tilde{t}_j R_{j+1}(u) = s.$$

The Tau–function has many properties, which can be found in textbooks and classical works on integrable systems [8, 50, 60, 61], but which are beyond the scope of the present book. In some sense, the Tau–function is the most fundamental function characterizing an integrable system, it contains all the information about the integrable system.

Here, for the integrable system satisfied by the $(2m+1, 2)$ minimal model, the Tau function can be computed by integrating twice the function $u(s)$ solution of a Painlevé type equation.

4.9 Large N limit

Our goal is to compare the minimal model's Tau function with the generating function of large maps introduced in section 1.3, which is by definition a formal power series of $1/N$, $\ln \tilde{Z} = \sum_g N^{2-2g}\, \tilde{F}_g$, where $\tilde{F}_g$ is the asymptotic generating function of large maps. The conjecture of topological gravity (proved below) is that:

$$\tau(s) \quad \overset{?}{=} \quad \tilde{Z}$$

Therefore, we need to study the formal large N expansion of the minimal model (p, q).

The large N limit for minimal models, is also called "dispersionless" limit. The parameter $1/N$, which we introduced as the coefficient of the identity in the commutator $[P, Q] = \frac{1}{N}\mathrm{Id}$, is called the **"dispersion" parameter**. In the large N limit P and Q tend to commute, $1/N$ plays the role of $\hbar$ in quantum mechanics, and the large N limit is a "classical limit".

Intuitively, in this limit, the operators P and Q will be replaced by functions, also the operator d will be replaced by a function z, and thus P and Q will be replaced by some functions of z and s.

Taking those observations as a guideline, in analogy with $Q = d^2 - 2u(s)$, and $P = d^p + \ldots$, we define:

Definition 4.10 *We define two functions $x(z, s)$ and $y(z, s)$ (which will be, as we shall see later, in some sense the large N limit of Q and P), polynomials in z, of respective degree 2 and p, of the form:*

$$x(z, s) = z^2 - 2u_0(s) \qquad , \qquad y(z, s) = z^p + O(z^{p-2})\ .$$

which we require to satisfy the following Poisson bracket equation (the "classical limit" of the string equation $[P, Q] = 1/N$):

$$\{y, x\} \overset{\text{def}}{=} \frac{\partial y}{\partial z}\frac{\partial x}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial z} = 1\,. \tag{V-4-24}$$

Proposition 4.4 *the general solution of this Poisson equation is:*

$$\begin{aligned} x(z, s) &= z^2 - 2u_0(s) \\ y(z, s) &= \sum_{j=0}^{m} \tilde{t}_j \left(z^{2j+1} \left(1 - \frac{2u_0(s)}{z^2}\right)^{j+1/2} \right)_+ + \sum_{j=0}^{m-1} c_j\, x(z, s)^j, \end{aligned}$$

$$= \sum_{j=0}^{m} \tilde{t}_j\, Q_j(z) + \sum_{j=0}^{m-1} c_j\, x(z,s)^j, \qquad \text{(V-4-25)}$$

where $()_+$ means the positive part of the large z Laurent series expansion, where $Q_j(z)$ was introduced in eq.(V-2-3), and where the function $u_0(s)$ has to satisfy the algebraic equation

$$\mathcal{P}(u_0(s)) = \sum_{j=0}^{m} \tilde{t}_j\, (-u_0(s)/2)^{j+1}\, \frac{(2j+1)!}{j!\,(j+1)!} = \frac{s}{4}. \qquad \text{(V-4-26)}$$

From now on, we shall always consider $c_j = 0$.

proof:

It is very similar to the proof of lemma 4.2, we leave it as an exercise for the reader.

We just mention that once we have seen that the function $y(z,s)$ must be of the form eq.(V-4-25), the Poisson equation $\{y, x\} = 1$, written at $z = 0$ reduces to:

$$\dot{u}_0(s)\, y'(0,s) = \frac{-1}{2},$$

i.e.

$$\sum_{j=0}^{m} \tilde{t}_j\, \dot{u}_0(s)\, (-u_0(s)/2)^j\, \frac{(2j+1)!}{(j!)^2} = -\frac{1}{2}$$

which can be integrated with respect to s and gives a polynomial equation for $u_0(s)$:

$$\mathcal{P}(u_0(s)) = \sum_{j=0}^{m} \tilde{t}_j\, (-u_0(s)/2)^{j+1}\, \frac{(2j+1)!}{j!\,(j+1)!} = \frac{s}{4}$$

which is clearly the classical limit of eq.(V-4-5) (i.e. it coincides with eq.(V-4-5) by removing all derivative terms). In other words, formally in the classical limit, the non-linear differential equation eq.(V-4-5) for $u(t)$, becomes an algebraic equation for $u_0(s)$.

This is the same equation which we encountered for large maps in eq.(V-3-4).

$\square$

For example, for pure gravity $m = 1$ we have the classical limit of eq.(V-4-7):

$$4\mathcal{P}(u_0) = 3\, u_0^2 - 2\tilde{t}_0\, u_0 = s. \qquad \text{(V-4-27)}$$

4.10 Topological expansion

In order to compare minimal models with large maps, we now look for a function $u(s)$ which is a formal series in $1/N$.

Proposition 4.5 *The formal series in $1/N$ solution $u(s)$ to the string equation eq.(V-4-5), can be expanded as an N^{-2} power series starting with u_0 (solution of $\mathcal{P}(u_0) = s/4$) as a leading order:*

$$u(s) = u_0(s) + \sum_k N^{-2k}\, u_k(s)$$

and where all coefficients u_k are rational functions of u_0 (their denominator is a power of $\mathcal{P}'(u_0)$):

$$u_k \in \mathbb{C}(u_0).$$

proof:

One notices that the string equation eq.(V-4-5) involves only N^2 and therefore the expansion is in powers of N^2 instead of N. Almost by definition of u_0, we see that $u_0(s)$ satisfies the string equation eq.(V-4-5) at $N=\infty$, and therefore is the first term of $u(s)$.

Since

$$\mathcal{P}(u_0) = s/4$$

we have

$$\dot{u}_0 = \frac{1}{4\mathcal{P}'(u_0)} \quad , \ \ddot{u}_0 = \frac{-\mathcal{P}''(u_0)}{16\,(\mathcal{P}'(u_0))^3} \quad , \ \dddot{u}_0 = \frac{3\mathcal{P}''(u_0)^2 - \mathcal{P}'(u_0)\mathcal{P}'''(u_0)}{64\,(\mathcal{P}'(u_0))^5} \quad , \ \dots$$

and in general, any derivative of u_0 with respect to s can be written as a rational function of u_0, whose denominator is a power of $\mathcal{P}'(u_0)$. Solving the string equation recursively involves derivatives of u_0, and thus each u_k is a rational function of u_0 whose denominator is a power of $\mathcal{P}'(u_0)$.

$\square$

Using the expression of Gelfand-Dikii polynomials eq.(V-4-4), the equation satisfied by u to order $O(1/N^4)$ is

$$\frac{s}{4} = \mathcal{P}(u) - \frac{\ddot{u}}{12\,N^2}\,\mathcal{P}''(u) - \frac{\dot{u}^2}{24\,N^2}\,\mathcal{P}'''(u) + O(1/N^4)$$

and thus we get

$$u_1 = \frac{\ddot{u}_0}{12}\,\frac{\mathcal{P}''(u_0)}{\mathcal{P}'(u_0)} + \frac{\dot{u}_0^2}{24}\,\frac{\mathcal{P}'''(u_0)}{\mathcal{P}'(u_0)} = \frac{1}{24}\,\left(\frac{\ddot{u}_0^2}{\dot{u}_0^2} - \frac{\dddot{u}_0}{\dot{u}_0}\right)$$

We could easily obtain $u_2, u_3, \dots$ by expanding to further orders.

Topological expansion for the Tau-function

Proposition 4.6 *We have:*

From the $1/N^2$ expansion of $u(s)$, we get that the Free energy $\mathcal{F}(s) = \ln\tau(s)$ such that $u = \frac{1}{N^2}\,\ddot{\mathcal{F}}$, also has a $1/N^2$ expansion:

$$\ln\tau = \mathcal{F} = \sum_{g=0}^{\infty} N^{2-2g}\mathcal{F}_g(u_0) \qquad , \qquad \ddot{\mathcal{F}}_g = u_g. \tag{V-4-28}$$

In particular we have

$$\mathcal{F}_0 = -4\,\sum_{j,k} \tilde{t}_j\,\tilde{t}_k\,\,(-u_0/2)^{k+j+3}\,\frac{j+k+4}{j+k+3}\,\frac{(2j+2)!}{j!\,(j+2)!}\,\frac{(2k+2)!}{k!\,(k+1)!}$$

$$\mathcal{F}_1 = \frac{-1}{24}\,\ln\left(-2\,\dot{u}_0\right) = \frac{1}{24}\,\ln\left(y'(0,s)\right).$$

proof:

We propose it as an exercise at the end of this chapter. $\mathcal{F}_1$ can be easily derived from the expression of u_1 above, and for $\mathcal{F}_0$, see the hints in the exercise.

$\square$

Topological expansion for the differential systems

Since the coefficients of the Lax matrix $\mathcal{D}(x,s)$ depend on $u(s)$ and its derivatives, it has a formal $1/N$ expansion:

$$\mathcal{D}(x,s) = \begin{pmatrix} A(x,s) & B(x,s) \\ C(x,s) & -A(x,s) \end{pmatrix} = \sum_g N^{-k} \mathcal{D}^{(k)}(x,s)$$

where

$$\begin{aligned} B(x,s) &= \sum_k N^{-2k}\, B_{2k}(x,s) \\ C(x,s) &= (z^2+2u-2u_0)B(x,s) - \frac{1}{2N^2}\,\ddot{B}(x,s) = \sum_k N^{-2k}\, C_{2k}(x,s) \\ A(x,s) &= \frac{-1}{2N}\,\dot{B}(x,s) = \sum_k N^{-2k-1}\, A_{2k+1}(x,s), \end{aligned}$$

and notice that B_{2k}, and thus C_{2k} and A_{2k+1} are polynomials of x, i.e. polynomials of $z^2 = x+2u_0$.

To leading order we have:

$$\mathcal{D}^{(0)}(x,s) = \begin{pmatrix} 0 & \overline{B}(x,u_0) \\ (x+2u_0)\,\overline{B}(x,u_0) & 0 \end{pmatrix} \tag{V-4-29}$$

$$\overline{B}(x,u_0) = \sum_{j=0}^{m}\sum_{k=0}^{j} \tilde{t}_j x^{j-k}\, u_0^k\, \frac{(-1)^k\,(2k-1)!!}{k!}$$

The determinant of $\mathcal{D}^{(0)}(x,s)$ is:

$$\det \mathcal{D}^{(0)}(x,s) = -\left(z\,\overline{B}(z^2-2u_0,u_0)\right)^2.$$

This means that the eigenvalues of $\mathcal{D}^{(0)}(x,s)$ are $\pm z\,\overline{B}(z^2-2u_0,u_0)$.

Notice that $z\,\overline{B}(z^2-2u_0,u_0)$ is precisely the function $y(z,s)$ of proposition. 4.4, in eq.(V-4-25).

Definition 4.11 *The "classical spectral curve" is the eigenvalue locus of the classical limit $\mathcal{D}^{(0)}(x)$ of the Lax matrix.*

If we parametrize x as $x = z^2 - 2u_0$, the eigenvalues of $\mathcal{D}^{(0)}(x,s))$ are:

$$y = \pm\, y(z,s)$$

where $y(z,s)$ is the function defined in eq.(V-4-25).

Written in a parametric form where $u_0 = u_0(s)$, the classical spectral curve is thus:

$$\mathcal{E}_{(2m+1,2)} = \left\{ \begin{array}{l} x(z,s) = z^2 - 2u_0 \\ y(z,s) = \sum_j \tilde{t}_j Q_j(z) = \sum_j \sum_l \tilde{t}_j \, z^{2j+1-2l} \, (-u_0/2)^l \ \frac{(2j+1)!}{j!} \, \frac{(j-l)!}{l!\,(2j+1-2l)!} \end{array} \right. \tag{V-4-30}$$

Remark 4.5 It is important to notice that it is a genus 0 hyperelliptical curve, which is equivalent to saying that it can be parametrized by a complex variable z (higher genus would be parametrized by a variable z living on a Riemann surface), and which is equivalent to saying that the polynomial y^2, written as a polynomial in x, has only one simple zero, located at $x = -2u_0$, all the other zeroes are double zeroes:

$$y^2 = z^2 \, (\overline{B}(x,u_0))^2 = (x + 2u_0) \, (\overline{B}(x,u_0))^2.$$

Remark 4.6 It is also the same curve as the blown up spectral curve considered in section 2. This is of course not an accident, this is an indication that indeed, large maps are related to the Tau-function of the $(p,2)$ minimal model. Our goal is to show that not only the large N limits coincide, but the full expansion.

4.11 WKB expansion

Similarly, we can look for a formal large N asymptotic expansion of the solutions $\psi(x,s)$ of the differential system. To leading order, it takes the WKB form:

$$\psi(x,s) \sim \frac{\mathrm{e}^{-N\int_{-2u_0}^{x} ydx}}{\sqrt{2}\,(-x-2u_0)^{\frac{1}{4}}} \left(1 + \sum_k N^{-k}\psi_k(x,s)\right)$$

$$\tilde{\psi}(x,s) \sim \frac{1}{\sqrt{2}}\,\mathrm{e}^{-N\int_{-2u_0}^{x} ydx}\ (x+2u_0)^{\frac{1}{4}} \left(1 + \sum_k N^{-k}\tilde{\psi}_k(x,s)\right)$$

and we recall that $z = (x+2u_0)^{\frac{1}{2}}$. The BKW expansion of the other solutions ϕ and $\tilde{\phi}$, are obtained by changing $N \to -N$. For the matrix Ψ, we have:

$$\Psi(x,s) \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{z}} & -\frac{1}{\sqrt{z}} \\ \sqrt{z} & \sqrt{z} \end{pmatrix} \ \hat{\Psi}(x,s)\ \mathrm{e}^{-N\,\sigma_3\int_{-2u_0}^{x} ydx}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ (denoted σ_3 because it is the 3^{rd} Pauli matrix), and

$$\hat{\Psi}(x,s) = \mathrm{Id}\ + \sum_{k=1}^{\infty} N^{-k}\,\Psi_k(x,s).$$

where each $\Psi_k(x,s)$ is a square matrix independent of N:

$$\Psi_k(x,s) = \begin{pmatrix} \psi_k(x,s) & \phi_k(x,s) \\ \tilde{\psi}_k(x,s) & \tilde{\phi}_k(x,s) \end{pmatrix}.$$

The fact that Ψ satisfies the differential systems $\Psi' = -N\,\mathcal{D}\,\Psi$ and $\dot\Psi = N\mathcal{R}\Psi$ imply for $\hat\Psi$:

$$\hat\Psi' = Ny\,\hat\Psi\,\sigma_3 - \frac{N}{2z}\begin{pmatrix} Bz^2+C & Bz^2-C-2Az \\ C-Bz^2-2Az & -Bz^2-C\end{pmatrix}\hat\Psi - \frac{1}{4z^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat\Psi$$

$$\dot{\hat\Psi} = -\,Nz\,\hat\Psi\,\sigma_3 + \frac{N}{z}\begin{pmatrix} z^2+u-u_0 & u_0-u \\ u-u_0 & -z^2+u_0-u\end{pmatrix}\hat\Psi - \frac{\dot u_0}{2z^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat\Psi$$

Let us expand it into powers of N, we have:

$$\begin{aligned}
B(x,s) &= \sum_k N^{-2k}\,B_{2k}(x,s)\\
C(x,s) &= (z^2+2u-2u_0)B(x,s) - \frac{1}{2N^2}\,\ddot B(x,s) = \sum_k N^{-2k}\,C_{2k}(x,s)\\
A(x,s) &= \frac{-1}{2N}\,\dot B(x,s) = \sum_k N^{-2k-1}\,A_{2k+1}(x,s),
\end{aligned}$$

and notice that B_{2k}, and thus C_{2k} and A_{2k+1} are polynomials of x, i.e. polynomials of z^2. Notice that

$$C_0(x,s) = z^2\,B_0(x,s) = z\,y.$$

that gives

$$\begin{aligned}
\psi_k' &= -\frac{1}{2z}\sum_{j\geq 1}(z^2B_{2j}+C_{2j})\psi_{k+1-2j} - \frac{1}{2z}\sum_{j\geq 1}(z^2B_{2j}-C_{2j})\tilde\psi_{k+1-2j}\\
&\quad + \sum_{j\geq 0}A_{2j+1}\tilde\psi_{k-2j} - \frac{1}{4z^2}\tilde\psi_k\\
\tilde\psi_k' &= 2y\tilde\psi_{k+1} + \sum_{j\geq 0}A_{2j+1}\psi_{k-2j} + \frac{1}{2z}\sum_{j\geq 1}(z^2B_{2j}-C_{2j})\psi_{k+1-2j}\\
&\quad + \frac{1}{2z}\sum_{j\geq 1}(z^2B_{2j}+C_{2j})\tilde\psi_{k+1-2j} - \frac{1}{4z^2}\psi_k\\
\dot\psi_k &= \frac{1}{z}\sum_{j\geq 1}u_j\,(\psi_{k+1-2j}-\tilde\psi_{k+1-2j}) - \frac{\dot u_0}{2z^2}\tilde\psi_k\\
\dot{\tilde\psi}_k &= -2z\tilde\psi_{k+1} + \frac{1}{z}\sum_{j\geq 1}u_j\,(\psi_{k+1-2j}-\tilde\psi_{k+1-2j}) - \frac{\dot u_0}{2z^2}\psi_k
\end{aligned} \tag{V-4-31}$$

and we have similar equations for ϕ_k and $\tilde\phi_k$:

We have the following Lemma:

Lemma 4.4 *$\forall\, k\geq 0$, $\psi_k(x,s)-\delta_{k,0}$ and $\tilde\psi_k(x,s)$ are polynomials of $1/z$ of the same parity as k and which behave like $O(1/z)$ at large z.*

proof:

We proceed by recursion. We have $\psi_0 = 1$ and $\tilde\psi_0 = 0$, so the recursion hypothesis holds for $k=0$.

Assume the recursion hypothesis at rank k.

Since $\dot{z} = \dot{u}_0/z$, we have that $\dot{\tilde{\psi}}_k$ is a polynomial of $1/z$, and thus from the 4th equation of eq.(V-4-31), that

$$\tilde{\psi}_{k+1} = \frac{1}{z^2}\Big(\text{Polynomial of } 1/z\Big)$$

i.e. $z^2\tilde{\psi}_{k+1}$ is also a polynomial in $1/z$, and it has the parity of $k+1$.

Then, the 1st equation of eq.(V-4-31) written at rank $k+1$ implies that ψ'_{k+1} is a Laurent polynomial of $1/z$ of parity $k+1$ (remember that B_{2j}, C_{2j}, A_{2j+1} are polynomials of z^2 and thus contain positive powers of z). After integrating with respect to $x = z^2 - 2u_0$, this implies that ψ_{k+1} must be a Laurent polynomial of $1/z$ of parity $k+1$, plus possibly a term proportional to $\ln z$ when $k+1$ is even:

$$\psi_{k+1} = \sum_{j\geq 0} a_{k+1,j} z^j + \ c_{k+1} \ln z + \sum_{j\geq 1} b_{k+1,j} z^{-j}.$$

However, from the large x behavior eq.(V-4-23) we know that at large z, we must have $\psi_{k+1}(z) = o(1)$ and thus the Log term must vanish, and thus $z\psi_{k+1}$ is a polynomial in $1/z$, and the parity is clearly k. We have proved the recursion hypothesis to rank $k+1$.

□

Examples:

to the first few orders

$$\psi_0 = 1 \qquad , \qquad \tilde{\psi}_0 = 0$$

$$\psi_1 = -\frac{1}{24}\,(\frac{\ddot{u}_0}{\dot{u}_0\, z} + \frac{\dot{u}_0}{z^3}) \qquad , \qquad \tilde{\psi}_1 = -\frac{\dot{u}_0}{4z^3}$$

Topological expansion of the kernel

The Christoffel Darboux kernel $K(x_1, x_2)$ can be rewritten as:

$$K(x_1,x_2) = \frac{\mathrm{e}^{-N\int_{z_2}^{z_1} y dx}}{2\sqrt{z_1 z_2}}\,\Big(\frac{\hat{\psi}(z_1)\hat{\tilde{\phi}}(z_2) - \hat{\tilde{\psi}}(z_1)\hat{\phi}(z_2)}{z_1 - z_2} + \frac{\hat{\tilde{\psi}}(z_1)\hat{\tilde{\phi}}(z_2) - \hat{\psi}(z_1)\hat{\phi}(z_2)}{z_1 + z_2}\Big),$$

and since each term has an expansion in $1/N$, whose coefficients are polynomials of $1/z_1$ and $1/z_2$, we have:

$$K(x_1,x_2) = \frac{\mathrm{e}^{-N\int_{z_2}^{z_1} y dx}}{2\sqrt{z_1 z_2}}\,\left(\frac{1}{z_1 - z_2} + \sum_{k=1}^{\infty} N^{-k} K_k(x_1,x_2)\right)$$

where each $K_k(x_1, x_2)$ is a polynomial in $1/z_1$ and in $1/z_2$.

This implies that the correlators also have a $1/N$ expansion:

$$\hat{W}_1(x) = -N\,y + \frac{1}{2z}\,\sum_{k=1}^{\infty} N^{-k}\, K_k(x,x).$$

$$\hat{W}_2(x_1,x_2) = \frac{1}{4z_1 z_2\,(z_1 - z_2)^2} - \frac{1}{(x_1 - x_2)^2} + O(N^{-1})$$

Topological expansion of the projectors $M(x)$

The projector $M(x)$ defined in eq.(4.7) also has a large N expansion:

$$M(x)=\sum_k N^{-k}M^{(k)}(x)=\frac{1}{2}\,\mathrm{Id}-\frac{1}{2}\begin{pmatrix}0 & 1/z\\ z & 0\end{pmatrix}+O(1/N)$$

Notice that we have

$$\forall\, x_1,x_2\,,\qquad \left[\frac{M^{(0)}(x_2)}{x_1-x_2}+\frac{1}{2z_2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\,,\,M^{(0)}(x_1)\right]=0$$

and thus

$$\forall\, x_1,x_2\,,\qquad \left[\frac{M(x_2)}{x_1-x_2}+\begin{pmatrix}0&0\\ \frac{1}{2z_2}&0\end{pmatrix}\,,\,M(x_1)\right]=O(1/N)$$

Lemma 4.5 (Topological expansion) *$N^{n-2}\hat{W}_n$ is a formal power series in powers of $1/N^2$*

$$\hat{W}_n(x_1,\dots,x_n)=\sum_{g=0}^{\infty}N^{2-2g-n}\,\hat{W}_n^{(g)}(x_1,\dots,x_n)$$

where each $\hat{W}_n^{(g)}$ is a rational function of the $z_i=\sqrt{x_i+2u_0}$, with poles only at $z_i=0$, except $\hat{W}_2^{(0)}$ and $\hat{W}_1^{(0)}$ which are:

$$\hat{W}_1^{(0)}=-\,y(z,s)$$

$$\hat{W}_2^{(0)}=\frac{1}{4z_1z_2}\,\frac{1}{(z_1-z_2)^2}\;-\frac{1}{(z_1^2-z_2^2)^2}=\frac{1}{4z_1z_2(z_1+z_2)^2}.$$

This Lemma makes some non-trivial claims, first that there is no odd power of $1/N$, second that $\hat{W}_n$ starts as N^{2-n}, and third that the coefficients are polynomials of $1/z_i$.
proof:

Notice that in the products $\prod_i K(z_{\sigma(i)},z_{\sigma(i+1)})$, all the exponentials cancel, and the square roots $1/\sqrt{z_i}$ appear only by pairs, so the result is, order by order in N^{-k}, a rational fraction of the z_i's having poles at $z_i=0$, or possibly at $z_i=z_j$. Except for $\hat{W}_1^{(0)}$ and $\hat{W}_2^{(0)}$, the poles at $z_i=z_j$ are at most simple poles, and it is easy to see that in the sum over permutations, the residues cancel, therefore there is no pole at $z_i=z_j$. Thus each $\hat{W}_n^{(g)}$ is a rational function of the z_i's having poles only at $z_i=0$. The cases of $\hat{W}_2$ and $\hat{W}_1$ need to be treated separately, and are easy.

The fact that $\hat{W}_n$ has a $1/N^2$ expansion instead of $1/N$ comes from a simple symmetry argument. In the expression of $\hat{W}_n$, changing $\psi\to\phi$ and $\tilde{\psi}\to\tilde{\phi}$, can also be obtained by permuting the x_i's, and since we take a symmetric sum, only the terms which are invariant under the exchange $\psi\to\phi$ and $\tilde{\psi}\to\tilde{\phi}$ contribute to $\hat{W}_n$. Exchanging the two solutions $\psi\to\phi$ and $\tilde{\psi}\to\tilde{\phi}$, is also equivalent to changing $N\to -N$, and therefore $\hat{W}_n$ has the parity $(-1)^n$, in N.

It remains to prove that the leading order is N^{2-n}. This is obvious for $n=1$ or $n=2$. For $n\geq 3$, we shall proceed by induction, by applying the insertion operator defined in section 4.6, which has the property that

$$\delta_{x_{n+1}}\hat{W}_n(x_1,\dots,x_n)=\hat{W}_{n+1}(x_1,\dots,x_n,x_{n+1}).$$

Let us write:

$$M(x)=xU(x)+A(x)-\frac{1}{N}\,\frac{d\,M(x)}{ds}\,\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

where

$$U(x)=\psi(x)\phi(x)\begin{pmatrix}0&0\\1&0\end{pmatrix}\qquad,\qquad A(x)=\psi(x)\tilde{\phi}(x)\,\mathrm{Id}+\psi(x)\phi(x)\begin{pmatrix}0&1\\2u&0\end{pmatrix}.$$

Observe that

$$\forall\, x,y,\alpha\ ,\qquad [A(x)+\alpha U(x),A(y)+\alpha U(y)]=0.$$

This implies that the insertion operator δ_y acts on $M(x)$ like

$$\begin{aligned}\delta_y M(x) &= [\frac{M(y)}{x-y}+U(y),M(x)]\\ &= -\frac{1}{N}\,\frac{1}{x-y}\left([xU(y)+A(y),\dot{M}(x)\begin{pmatrix}0&0\\1&0\end{pmatrix}]-[xU(x)+A(x),\dot{M}(y)\begin{pmatrix}0&0\\1&0\end{pmatrix}]\right)\\ &\quad+\frac{1}{N^2}\,\frac{1}{x-y}\left(\dot{M}(x),[\dot{M}(y),\begin{pmatrix}0&0\\1&0\end{pmatrix}]-\dot{M}(y)[\dot{M}(x),\begin{pmatrix}0&0\\1&0\end{pmatrix}]\right)\begin{pmatrix}0&0\\1&0\end{pmatrix}\end{aligned}$$

and it acts on $u(s)$ by eq.(V-4-20), i.e.

$$\delta_y u(s)=\frac{1}{N}\,\frac{d}{ds}\,\psi(y)\phi(y)$$

therefore the action of the operator δ_y brings a factor $1/N$, and the result is again expressed in terms of $M(x)$, $M(y)$, and $u(s)$ and their d/ds derivatives, and we recall that the insertion operator commutes with d/ds.

Since

$$\hat{W}_2(x_1,x_2)=\frac{\mathrm{Tr}\,M(x_1)M(x_2)}{(x_1-x_2)^2}-\frac{1}{(x_1-x_2)^2}$$

is of order $O(1)$ and is expressed only in terms of M, and since for $n\geq 3$

$$\hat{W}_n(x_1,\dots,x_n)=\delta_{x_n}\,\hat{W}_{n-1}(x_1,\dots,x_{n-1})$$

we see that by recursion:

$$\hat{W}_n=O(N^{2-n}).$$

□

We mention that this theorem is far from being true for any Lax matrix. It holds because our Lax matrix is related to the $(p,2)$ minimal model.

4.12 Link with symplectic invariants

We have found that the minimal model correlators $\hat{W}_n$ have a formal large N expansion of the form

$$\hat{W}_n(x_1,\dots,x_n) = \sum_g N^{2-2g-n}\,\hat{W}_n^{(g)}(x_1,\dots,x_n)$$

where each $\hat{W}_n^{(g)}$ with $2-2g-n<0$ is a rational function of the $z_i=\sqrt{x_i+2u_0}$, with poles only at $z_i=0$. And we have found that they satisfy loop equations in theorem 4.2.

Let us define:

$$\hat{\omega}_n^{(g)}(z_1,\dots,z_n) = \hat{W}_n^{(g)}(x(z_1),\dots,x(z_n))\prod_{i=1}^n x'(z_i) + \frac{\delta_{n,2}\delta_{g,0}x'(z_1)x'(z_2)}{(x(z_1)-x(z_2))^2}$$

The first few are easily computed from the BKW expansion, and one finds:

$$\hat{\omega}_1^{(0)}(z) = -\,y(z,s)\,x'(z)$$

$$\hat{\omega}_2^{(0)}(z_1,z_2) = \frac{1}{(z_1-z_2)^2}$$

and all the other $\hat{\omega}_n^{(g)}(z_1,\dots,z_n)$ with $2-2g-n<0$ are symmetric polynomials of $1/z_i$.

Then, since they satisfy loop equations, we have:

Theorem 4.4 *The $\hat{\omega}_n^{(g)}$ can be computed by the "topological recursion"*

$$\begin{aligned}\hat{\omega}_{n+1}^{(g)}(z_0,z_1,\dots,z_n) &= -\operatorname*{Res}_{z\to 0}\frac{\frac{1}{z_0-z}-\frac{1}{z_0+z}}{2y(z,t)\,x'(z)}\Big[\hat{\omega}_{n+2}^{(g-1)}(z,-z,z_1,\dots,z_n)\\ &\quad+\sum_{h=0}^{g}\sideset{}{'}\sum_{I\subset\{z_1,\dots,z_n\}}\hat{\omega}_{1+\#I}^{(h)}(z,I)\,\hat{\omega}_{1+n-\#I}^{(g-h)}(-z,\{z_1,\dots,z_n\}\setminus I)\Big]\end{aligned}$$

In other words, the differentials $\hat{\omega}_n^{(g)}(z_1,\dots,z_n)\prod_i dz_i$, are the symplectic invariant correlators for the spectral curve of eq.(V-4-30) (see chapter VII for the definition of symplectic invariants of the spectral curve).

proof:

Notice that, since $\hat{\omega}_{n+1}^{(g)}(z_0,z_1,\dots,z_n)$ is a polynomial in $1/z_0$, we have the Cauchy identity:

$$\begin{aligned}\hat{\omega}_{n+1}^{(g)}(z_0,z_1,\dots,z_n) &= -\operatorname*{Res}_{z\to z_0}\frac{dz}{z_0-z}\,\hat{\omega}_{n+1}^{(g)}(z,z_1,\dots,z_n)\\ &= \operatorname*{Res}_{z\to 0}\frac{dz}{z_0-z}\,\hat{\omega}_{n+1}^{(g)}(z,z_1,\dots,z_n)\\ &= -\operatorname*{Res}_{z\to 0}\frac{dz}{(z_0-z)y(z)x'(z)}\,2\hat{\omega}_1^{(0)}(z)\,\hat{\omega}_{n+1}^{(g)}(z,z_1,\dots,z_n)\end{aligned}$$

Then, the loop equations (theorem 4.2) imply that the quantity

$$2\hat{\omega}_1^{(0)}(z)\,\hat{\omega}_{n+1}^{(g)}(z,z_1,\ldots,z_n)+\sum_{h=0}^{g}\sum_{I\subset\{z_1,\ldots,z_n\}}^{\prime}\hat{\omega}_{1+\#I}^{(h)}(z,I)\,\hat{\omega}_{1+n-\#I}^{(g-h)}(z,\{z_1,\ldots,z_n\}\setminus I)$$
$$+\hat{\omega}_{n+2}^{(g-1)}(z,z,z_1,\ldots,z_n)$$

is equal to $x'(z)^2$ times a rational function of $x(z)$, with no pole at $z=0$ (in fact it is a polynomial of $x(z)$ plus a rational function of $x(z)$ with poles at $z=\pm z_i$), in other words it cannot contribute to the residue. This shows that

$$\begin{aligned}\hat{\omega}_{n+1}^{(g)}(z_0,z_1,\ldots,z_n) &= \operatorname*{Res}_{z\to 0}\frac{1}{(z_0-z)\,y(z,t)\,x'(z)}\Big[\hat{\omega}_{n+2}^{(g-1)}(z,z,z_1,\ldots,z_n)\\ &\quad +\sum_{h=0}^{g}\sum_{I\subset\{z_1,\ldots,z_n\}}^{\prime}\hat{\omega}_{1+\#I}^{(h)}(z,I)\,\hat{\omega}_{1+n-\#I}^{(g-h)}(z,\{z_1,\ldots,z_n\}\setminus I)\Big]\end{aligned}$$

Then, using the fact that each $\hat{\omega}_n^{(g)}$ has a given parity in the z_i's, it is easy to complete the proof.

A special care is needed for $\hat{\omega}_1^{(1)}(z_0)$ because $\hat{\omega}_2^{(0)}(z,z)$ is ill-defined, but we leave to the reader to check that the theorem also holds for that case.

□

As an immediate consequence we have that:

Corollary 4.1 *The correlation function $\hat{\omega}_n^{(g)}$ of the minimal model $(2m+1,2)$, coincide with the generating function of large maps $\tilde{\omega}_n^{(g)}$ of genus g and $n\geq 1$ boundaries (defined in theorem 3.1):*

$$\hat{\omega}_n^{(g)}=\tilde{\omega}_n^{(g)}.$$

proof:

The topological recursion theorem 4.4 for the minimal model $(2m+1,2)$, is identical to the topological recursion of theorem 3.1 for the generating functions of large maps.

Therefore, the $\hat{\omega}_n^{(g)}$ of the minimal model $(2m+1,2)$ and the generating function of large maps $\tilde{\omega}_n^{(g)}$ are both equal to the symplectic invariants of the spectral curve $(x(z,s),y(z,s))$, they satisfy the same topological recursion with the same initial condition. □

4.13 Tau function

Here, we prove that the double scaling limits $\tilde{F}_g$ of the large maps generating functions (see section 1.3), which coincide with the symplectic invariants $\mathcal{F}_g$ of our spectral curve (see theorem 3.3), do also coincide with the coefficients of the topological expansion of the minimal model Tau-function introduced in section V.4.8, i.e. prop. 4.6, eq.(V-4-28):

$$\ln\tau=\sum_g N^{2-2g}\hat{F}_g \qquad,\ \partial^2\hat{F}_g/\partial s^2=u_g(s).$$

From the Poisson equation eq.(V-4-24), it is easy to see that our spectral curve has the property that

$$\left.\frac{\partial y(z,s)}{\partial s}\right|_{x(z,s)} = -\frac{1}{2z}$$

and thus

$$x'(z)\left.\frac{\partial}{\partial s}\right|_{x} y(z,s) = -1 = \operatorname*{Res}_{z'\to\infty} \frac{z'\,dz'}{(z-z')^2} = \operatorname*{Res}_{z'\to\infty} z'\,\hat\omega_2^{(0)}(z,z')\,dz'$$

Knowing that, it follows from general property of symplectic invariants $\mathcal{F}_g$ of a spectral curve (see chapter VII), that:

$$\frac{\partial}{\partial s}\tilde F_g = \operatorname*{Res}_{z\to\infty} z\,\hat\omega_1^{(g)}(z)\,dz = 2\operatorname*{Res}_{x\to\infty} \sqrt{x+2u_0}\;\hat W_1^{(g)}(x)\,dx$$

In other words

$$\frac{\partial}{\partial s}\tilde F_g = 2\operatorname*{Res}_{x\to\infty} \hat W_1^{(g)}(x)\,\sqrt{x}\,dx$$

and summing over g:

$$\frac{1}{N}\frac{\partial\tilde F}{\partial s} = 2\operatorname*{Res}_{x\to\infty} \hat W_1(x)\,\sqrt{x}\,dx = \frac{1}{N}\frac{\partial\hat F}{\partial s}$$

where the last equality holds by definition of the τ-function in section 4.8.

This proves:

Theorem 4.5 *Near a m^{th} order critical point, the coefficients of the double scaling limit of large maps $\tilde F_g$ such that $F_g\sim(t-t_c)^{(2-2g)\frac{2m+3}{2m+2}}\,\tilde F_g$, are the symplectic invariants of the classical spectral curve eq.(V-4-30), and are such that the generating series:*

$$\tau = \exp\sum_g N^{2-2g}\tilde F_g$$

is the Tau-function of the $(2m+1,2)$ minimal model, or also, $u(s)=d^2\ln\tau/ds^2$ satisfies the $m+1^{\text{th}}$ Gelfand Dikii equation:

$$R_{m+1}(u(s)) = s.$$

We have thus seen, that the asymptotic generating function which counts large maps near a critical point of order m, is the Tau-function for the $(2m+1,2)$ reduction of the KdV hierarchy. In particular, its second derivative satisfies the $(m+1)^{\text{th}}$ Gelfand-Dikii equation.

4.14 Large N and large s

Rescaling N

We have introduced the parameter N as a scaling parameter in order to define formal power series.

But notice that N is redundant, it can be absorbed by the change of variable $s = N^{-\frac{p+1}{p+2}}\,\tilde{s}$ and $u(s) = N^{\frac{-2}{p+2}}\,\tilde{u}(\tilde{s})$. We have

$$Q = N^{\frac{-2}{p+2}}\,\tilde{Q} \qquad , \qquad P = N^{\frac{-p}{p+2}}\,\tilde{P}$$

with

$$\tilde{Q} = \tilde{d}^2 - 2\tilde{u}(\tilde{s}) \qquad , \qquad \tilde{P} = \tilde{d}^p - p\tilde{u}\tilde{d}^{p-2} + \ldots \qquad , \qquad \tilde{d} = \frac{d}{d\tilde{s}},$$

and they satisfy the string equation without $1/N$:

$$[\tilde{P}, \tilde{Q}] = \mathrm{Id}.$$

Homogeneous Case

A case particularly interesting is when all $\tilde{t}_j$'s with $j < m$ vanish. In that case, the equation for $u_0(s)$ is simply:

$$\frac{s}{4} = \mathcal{P}(u_0) = \tilde{t}_m\, \frac{(2m+1)!}{m!\,(m+1)!}\, (-u_0/2)^{m+1},$$

i.e. $\mathcal{P}(u_0)$ is a homogeneous polynomial of u_0.

This implies that the BKW expansion of $u(s)$ has only homogeneous terms:

$$u(s) = u_0 + \sum_{g=1}^{\infty} N^{-2g}\; c_g\; u_0^{1-g(2m+3)}$$

where c_g are some complex coefficients.

Using the reparametrization $s = N^{-\frac{p+1}{p+2}}\,\tilde{s}$ and $u(s) = N^{\frac{-2}{p+2}}\,\tilde{u}(\tilde{s})$, this amounts to writing a large $\tilde{s}$ expansion for $\tilde{u}$:

$$\tilde{u}(\tilde{s}) = \sum_{g=0}^{\infty} \tilde{u}_g\; \tilde{s}^{\frac{2}{p+1}\,(1-g(p+2))} \qquad , \qquad \tilde{u}_g = c_g\, \left(\frac{-2\,\tilde{t}_m\; (-1)^m\; (2m+1)!!}{(m+1)!}\right)^{-\frac{2}{p+1}\,(1-g(p+2))}.$$

The coefficients $\tilde{u}_g$ can be found by inserting this expansion into the Gelfand Dikii equation $R_{m+1}(u) = s$, or also, since $F = \sum_g N^{2-2g}\,\tilde{F}_g(s)$ and $\ddot{F} = u(s)$, we have just shown that, for $g \geq 2$:

$$\frac{(1-g)\,(m+1)^2}{(2m+3)\,(m+2-g(2m+3))}\,\tilde{u}_g = \tilde{F}_g = \mathcal{F}_g(\{x(z,s), y(z,s)\}).$$

We thus formulate the theorem:

Theorem 4.6 *If $\tilde{u}(\tilde{s})$ writen as a large $\tilde{s}$ series*

$$\tilde{u}(\tilde{s}) = \sum_{g=0}^{\infty} \tilde{u}_g\; \tilde{s}^{\frac{2}{p+1}\,(1-g(p+2))}$$

is solution of the $m+1^{\text{th}}$ Gelfand Dikii equation (here we choose $N=1$)

$$\tilde{t}_m \; R_{m+1}(\tilde{u}) = \tilde{s},$$

then

$$(-\tilde{u}_0/2)^{m+1} = \frac{s}{4\,\tilde{t}_m}\;\frac{m!\,(m+1)!}{(2m+1)!}$$

$$\tilde{u}_1 = -\frac{m}{24\,(m+1)}$$

and for $g \geq 2$, the coefficients $\tilde{u}_g$ of the expansion, are related to the symplectic invariants $\mathcal{F}_g$ of the spectral curve $\tilde{\mathcal{S}}$

$$\tilde{\mathcal{S}} = \left\{ \begin{array}{l} x(z) = z^2 - 2\tilde{u}_0 \\ y(z) = \tilde{t}_m Q_m(z) = \tilde{t}_m \; \sum_{j=0}^{m} \; z^{2m+1-2j}\,(-\tilde{u}_0/2)^j \; \frac{(2m+1)!}{m!} \; \frac{(m-j)!}{j!\,(2m+1-2j)!} \end{array} \right.$$

as:

$$\frac{(1-g)\,(m+1)^2}{(2m+3)\,(m+2-g(2m+3))}\,\tilde{u}_g = \mathcal{F}_g(\tilde{\mathcal{S}}).$$

and as an immediate corollary:

Theorem 4.7 *Near a m^{th} order critical point, the double scaling limit of the generating functions of large maps of genus g:*

$$\tilde{F}_g = \lim_{\epsilon\to 0} \; \epsilon^{(2g-2)\frac{2m+3}{2m+2}} \; t^{2g-2}\, F_g$$

are related to the coefficients $\tilde{u}_g\tilde{()}$ of the large $\tilde{s}$ expansion of the solution of the $m+1^{\text{th}}$ Gelfand-Dikii equation:

$$\frac{(1-g)\,(m+1)^2}{(2m+3)\,(m+2-g(2m+3))}\,\tilde{u}_g = \tilde{F}_g.$$

This theorem is an indication that large maps are related to Liouville conformal quantum field theory coupled to the $(2m+1,2)$ minimal model.

4.15 Example: Pure gravity case

Let us illustrate all this on the important example of pure gravity case, $m=1$, the $(3,2)$ minimal model.

We have:

$$Q = d^2 - 2u \qquad , \qquad P = d^3 - 3ud - \frac{3}{2N^2}\,\dot{u}.$$

The string equation $[P,Q] = \frac{1}{N}\,\mathrm{Id}$ gives the Painlevé I equation for $u(s)$:

$$3u^2 - \frac{1}{2N^2}\,\ddot{u} = s.$$

There is a formal solution of this equation with an expansion in powers of $1/N^2$:

$$u(s) = -\sqrt{\frac{s}{3}} - \frac{1}{48\,N^2\,s^2} + \frac{49}{32\,N^4\,\sqrt{3}\,s^{\frac{9}{2}}} + O(1/N^6)$$

which can be written

$$u(s) = \sum_{g=0}^{\infty} c_g\ N^{-2g}\ u_0^{1-5g} \qquad , \qquad u_0 = -\sqrt{\frac{s}{3}}.$$

With the rescaling

$$u = N^{\frac{-2}{5}}\,\tilde{u} \qquad , \qquad s = N^{\frac{-4}{5}}\,\tilde{s}$$

we have

$$\tilde{u} = \sum_g \tilde{u}_g\ \tilde{s}^{\frac{1}{2}-\frac{5}{2}g}$$

The free energy $\mathcal{F}(s)$ such that $N^{-2}\ddot{\mathcal{F}} = u(s)$ has an expansion:

$$\mathcal{F}(s) = -\frac{4}{15\sqrt{3}}\,N^2\,s^{\frac{5}{2}} + \frac{\ln s}{48} + \frac{7}{40\sqrt{3}\,N^2\,s^{\frac{5}{2}}} + \sum_{g\geq 3}(N\,s^{\frac{5}{4}})^{2-2g}\ \tilde{F}_g$$

For example, the first few correlators computed from the topological recursion are

$$\tilde{\omega}_3^{(0)}(z_1,z_2,z_3) = \frac{1}{6u_0}\,\frac{dz_1}{z_1^2}\,\frac{dz_2}{z_2^2}\,\frac{dz_3}{z_3^2}$$

$$\tilde{\omega}_1^{(1)}(z) = \frac{dz}{3^2\ 2^4}\left(\frac{3}{z^4\,u_0} + \frac{1}{z^2\,u_0^2}\right).$$

5 Summary: large maps and Liouville gravity

We have seen that

• Large maps are obtained when the weights t_k of k–gons are tuned to some critical, or multi–critical values. At those critical values, the disc amplitude $W_1^{(0)}(x)$ has cusps of the form $W_1^{(0)}(x) \sim (x-a)^{p/q} \sim \frac{V'(x)}{2} + C\,(x-a)^{m+1/2}$.

• The tuning of the t_k's

$$t_k = t_{k,c} + \sum_j C_{k,j}\ (1-t/t_c)^{\nu\nu_j}\,\tilde{t}_j$$

comes with some critical exponents

$$\nu = \frac{1}{p+q-1} = \frac{1}{2m+2} \qquad , \qquad \nu_j = 2(m-j)$$

We then have the scalings

$$F_g(t,\{t_k\}) \sim (1-t/t_c)^{(2-2g)(1-\gamma/2)}\,t_c^{2-2g}\,\tilde{F}_g(\{\tilde{t}_j\})$$

with the exponent (called "**string susceptibility exponent**" by physicists)

$$\gamma = \frac{-2}{p+q-1} = \frac{-1}{m+1}.$$

Those exponents agree with the KPZ formula.

• The asymptotic generating functions of large maps, are obtained by the topological recursion, corresponding to the spectral curve:

$$\mathcal{E}_{(2m+1,2)} = \begin{cases} x(z,s) = z^2 - 2u_0 \\ y(z,s) = \sum_j \tilde{t}_j Q_j(z) = \sum_j \sum_l \tilde{t}_j\, z^{2j+1-2l}\, (-u_0/2)^l\ \frac{(2j+1)!}{j!}\, \frac{(j-l)!}{l!\,(2j+1-2l)!} \end{cases}$$

which is the blow up of the cusp singularity of $W_1^{(0)}(x) \sim (x-a)^{p/q}$

• This means that when $t \to t_c$

$$F_g \sim (1-t/t_c)^{(2-2g)\frac{2m+2}{2m+3}}\, t_c^{2-2g} \tilde{F}_g = (1-t/t_c)^{(2-2g)\frac{2m+2}{2m+3}}\, t_c^{2-2g}\, \mathcal{F}_g(\mathcal{E}_{(2m+1,2)}).$$

and $1 - t/t_c = \epsilon^2$ is the "mesh–size".

• The asymptotic generating functions of large maps, $\tilde{F}_g$, are such that

$$\tau = \mathrm{e}^{\sum_g N^{2-2g}\, \tilde{F}_g}$$

is the Tau–function of the m^{th} reduction of the KdV hierarchy of integrable equations, called the $(2m+1, 2)$ minimal model coupled to gravity.

• This means that the second derivative u of $\ln \tau$, satisfies a non-linear differential equation of Painlevé type, namely the $m+1^{\text{st}}$ Gelfand-Dikii equation:

$$R_{m+1}(u) = s.$$

• This means that the asymptotic generating functions of large maps coincide with those of the Liouville conformal field theory coupled to gravity.

6 Exercises

Exercise 1: Prove proposition 4.6, i.e. that

$$\mathcal{F}_0 = -4 \sum_{j,k} \tilde{t}_j\, \tilde{t}_k\, (-u_0/2)^{k+j+3}\, \frac{j+k+4}{j+k+3}\, \frac{(2j+2)!}{j!\,(j+2)!}\, \frac{(2k+2)!}{k!\,(k+1)!}.$$

Hint: first look for a polynomial $S(u_0)$ such that $\frac{d}{ds} S(u_0) = 4\, u_0$, and show that

$$S'(u_0) = 16 u_0\, \mathcal{P}'(u_0).$$

From there, and from the explicit expression of $\mathcal{P}(u_0)$, deduce $S(u_0)$.

Then look for a polynomial $\Xi(u_0)$, such that $\frac{d}{ds}\,\Xi(u_0) = S(u_0)$, and show that

$$\Xi'(u_0) = \mathcal{P}'(u_0)\ S(u_0).$$

From there, deduce the expression of $\Xi(u_0)$. It satisfies $d^2/ds^2\Xi = u_0$, and thus $\Xi = \mathcal{F}_0$.

Exercise 2: Prove lemma 4.1 and the recursion for the Gelfand-Dikii polynomials eq.(V-4-3).

Hint: To prove lemma 4.1, show that

$$[\Big((Q^{j-\frac{1}{2}})_+\Big)^2, Q] = (Q^{j-\frac{1}{2}})_+\,[(Q^{j-\frac{1}{2}})_+, Q] + [(Q^{j-\frac{1}{2}})_+, Q]\,(Q^{j-\frac{1}{2}})_+$$

is an operator of order at most $2j-1$, and this implies that $[(Q^{j-\frac{1}{2}})_+, Q]$ must be an operator of order 0, i.e. a function of s.

Using $(Q^{1/2})_+ = d$ find $R_1 = -2u$, and then proceed by recursion on j.

First show (using the recursion hypothesis) that it is possible to choose two functions $\alpha_j(s)$ and $\beta_j(s)$ such that

$$\Big(Q\,(Q^{j-\frac{1}{2}})_+ + \alpha_j d + \beta_j\Big)^2 = Q^{2j+1} + O(d^{2j})$$

i.e. that

$$(Q^{j+\frac{1}{2}})_+ = Q\,(Q^{j-\frac{1}{2}})_+ + \alpha_j d + \beta_j = (Q^{j-\frac{1}{2}})_+\,Q + (\alpha_j + \dot{R}_j)d + \beta_j$$

Then, writing that $[(Q^{j+\frac{1}{2}})_+, Q]$ must be an operator of degree 0, find the coefficients α_j, β_j, and find the recursion relation for R_j.