Chapter II

Formal matrix integrals

In this chapter we introduce the notion of a formal matrix integral, which is a very useful for combinatorics, as it turns out to be identical to the generating function of maps of chapter.I.

A formal integral is a formal series (an asymptotic series) whose coefficients are Gaussian matrix integrals, it is not necessarily convergent (in fact it is definitely not convergent, it has a vanishing radius of convergency).

Then, using Wick's theorem to compute Gaussian integrals in a combinatorial way, we relate formal matrix integrals to generating functions for maps.

The relationship between formal matrix integrals and maps, was first noticed by 'tHooft in the context of the study of strong nuclear interactions [**?**], and then really introduced as a tool for studying maps by Brezin-Itzykson-Parisi-Zuber in 1978 [**?**].

1 Definition of a formal matrix integral

1.1 Introductory example: 1-matrix model and quadrangulations

Consider the following polynomial moment of a gaussian integral over the set of hermitian $N \times N$ matrices:

$$A_k(N) = \frac{N^k}{k!\, 4^k} \int_{H_N} dM \ \ (\operatorname{Tr} M^4)^k \ \ \mathrm{e}^{-N \operatorname{Tr} \frac{M^2}{2}}$$

where M is a $N \times N$ hermitian matrix, with measure

$$dM = \frac{1}{2^N\, (\pi/N)^{N^2/2}} \prod_{i=1}^{N} dM_{ii} \ \prod_{i<j} d\mathrm{Re} M_{ij} \ d\mathrm{Im} M_{ij}$$

normalized so that $\int dM\ \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}} = 1$.

We shall see below that $A_k(N)$ is a polynomial in N and $1/N$, so it can be continued to any $N \in \mathbb{C}^*$.

With the sequence $A_k(N)$, $k = 0, 1, 2, \ldots, \infty$, we define a **formal power series** (asymptotic series) in powers of a variable which we choose to call t_4 because it is associated to $\mathrm{Tr}\, M^4$ (later we shall associate t_n to $\mathrm{Tr}\, M^n$):

$$Z_N(t_4) = \sum_{k=0}^{\infty} t_4^k\ A_k(N).$$

$Z_N(t_4)$ is well defined as a **formal power series** in t_4, in other words, $Z_N(t_4)$ is **nothing but a notation** which summarizes all the coefficients $A_k(N)$ in only one symbol $Z_N(t_4)$. This means that every time we are going to write properties or equations for $Z_N(t_4)$, we actually mean properties of the coefficients in the small t_4 expansion. Writing the equations in terms of $Z_N(t_4)$ is merely a shorter way of writing equations for $A_k(N)\ \forall k$.

We are never going to consider $Z_N(t_4)$ as a usual function of t_4, and in fact, for $t_4 > 0$ the series $Z_N(t_4)$ is never convergent (in the Borel sense for instance).

1.2 Comparison with convergent integrals

The definition of a formal matrix integral $Z_N(t_4)$ is not to be confused with the hermitean **convergent matrix integral**:

$$\begin{aligned} Z_{\mathrm{conv}}(t_4, N) &= \int_{H_N} dM\ \mathrm{e}^{-N\,\mathrm{Tr}\,(\frac{M^2}{2} - t_4\frac{M^4}{4})} \\ &= \int_{H_N} \sum_{k=0}^{\infty} t_4^k\ \frac{N^k}{k!\,4^k} dM \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}}\ (\mathrm{Tr}\, M^4)^k \end{aligned}$$

One should notice that $Z_{\mathrm{conv}}(t_4, N)$ is well defined only for $\mathrm{Re}(t_4) < 0$.

The existence and nature of large N asymptotics of hermitean convergent matrix integrals is a difficult problem which has been solved in a few cases, and which remains an open question in many cases at the time this book is being written (2-matrix model for instance).

The only difference in the definition of $Z_N(t_4)$ and $Z_{\mathrm{conv}}(t_4, N)$, is that the order of the sum over k and the integral over H_N has been exchanged. In general, the sum and the integral don't commute, and in general:

$$\boxed{Z_N(t_4)\ \neq\ Z_{\mathrm{conv}}(t_4, N)}$$

in other words:

$$\sum_{k=0}^{\infty} \int_{H_N} t_4^k\ \frac{N^k}{k!\,4^k} dM \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}}\ (\mathrm{Tr}\, M^4)^k \neq \int_{H_N} \sum_{k=0}^{\infty} t_4^k\ \frac{N^k}{k!\,4^k} dM \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}}\ (\mathrm{Tr}\, M^4)^k$$

Those two definitions of a matrix integral differ even after Borel resummation and analytical continuation from $t_4 > 0$ (which is the interesting regime for combinatorics) to $t_4 < 0$ (where $Z_{\text{conv}}(t_4)$ is well defined) they do not necessarily coincide, they might still differ by exponentialy small terms.

Convergent matrix integrals are not the topic of this book, and readers interested in asymptotic properties of large Matrix integrals, can refer to for instance [].

1.3 Formal integrals

So far, we have studied the example of a formal matrix integral with quartic potential, now let us give the general definition of a formal integral of the form:

$$\int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}\, dM.$$

The idea is to expand (Taylor series) the exponential of the non-quadratic part of $V(M)$, and write the integral as an infinite sum of polynomial moments of a gaussian integral, and **then invert the integral and the summation**.

More preciselly, let

$$V(M) = \frac{M^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} M^j$$

be called the potential, then we define the following polynomial moment of a Gaussian integral:

$$A_k = \frac{1}{k!}\frac{N^k}{t^k} \int_{H_N} dM\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr}\frac{M^2}{2}} \left(\sum_{j=3}^{d} \frac{t_j}{j} \operatorname{Tr} M^j\right)^k .$$

Lemma 1.1 *A_k is a polynomial in t such that:*

$$A_k = \sum_{m=k/2}^{[(d-2)k/2]} A_{k,m}\, t^m .$$

proof:

A monomial moment of a Gaussian integral vanishes if the degree of the monomial is odd, and is proportional to t to the power half the degree, if the degree is even. The polynomial $(\sum_{j=3}^{d} \frac{t_j}{j} \operatorname{Tr} M^j)^k$ can be decomposed into a finite sum of monomials in M of the form:

$$\prod_{j=3}^{d} (\operatorname{Tr} M^j)^{n_j} \qquad , \qquad \sum_{j=3}^{d} n_j = k$$

i.e. of degree $\sum_j j n_j$. Therefore such a term contributes to A_k with a power of t^m equal to:

$$m = -k + \frac{1}{2}\sum_{j=3}^{d} j n_j = \frac{1}{2}\sum_{j=3}^{d} (j-2) n_j \geq \frac{1}{2}\sum_{j=3}^{d} n_j = \frac{k}{2}.$$

The upper bound $m \leq (d-2)k/2$ is easily obtaind because $j \leq d$ and $n_j \leq k$. $\square$

This Lemma allows to define:

$$\tilde{A}_m = \sum_{k=0}^{2m} A_{k,m}.$$

Definition 1.1 *The formal integral is the formal power series in t:*

$$Z(t) = \int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} = \sum_{m=0}^{\infty} t^m \, \tilde{A}_m.$$

Remark 1.1 It is also a formal power series in each t_j with $3 \leq j \leq d$. We may choose $t = 1$ and expand in powers of t_3 or t_4..., as we did for quadrangulations. It is clear that t can be absorbed by a redefinition $M \to \sqrt{t}M$ and $t_j \to t^{\frac{j}{2}-1} t_j$, exactly like in section 2.3 of chapter I.

Remark 1.2 The formal integral and the convergent matrix integral differ by the order of integration and sum. In general the two operations do not commute, and the formal integral and the convergent integral are different:

$$\int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} \, dM \neq \int_{H_N} \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)} \, dM.$$

Remark 1.3 We shall see below in section II.2.2, that each $A_{k,m}$ is a Laurent polynomial in N:

$$A_{k,m} = \sum_{g=-g_{\min}(k,m)}^{g_{\max}(k,m)} N^g \, A_{k,m}^{(g)}$$

so that each $\tilde{A}_m$ is also a Laurent polynomial of N, and thus, to a given order t^m, the formal integral is a Laurent **polynomial in** N, and thus a formal matrix integral **always has a** $1/N$ **expansion**.

In other words, the question of a $1/N$ expansion is trivial for formal integrals, whereas it is a difficult question for convergent integrals (mostly unsolved for multi-matrix integrals with complex potentials).

Remark 1.4 Most of physicist's works in so-called **"2d-quantum-gravity"** are actually using that "formal" definition of a matrix integral (in fact almost all works in quantum field theory after Feynman's works, use formal integrals). Most of the works initiated by Brezin-Itzykson-Parisi-Zuber [**?**] in 1978 assume the formal definition of matrix integrals, and are correct and rigorous only with that definition, they are often wrong if one uses convergent hermitian matrix integrals instead.

2 Wick's theorem and combinatorics

2.1 Generalities about Wick's theorem

Wick's theorem is a very useful theorem for combinatorics. It gives a combinatoric way of computing Gaussian expectation values, or conversely, it gives an algebraic and analytical way of enumerating graphs.

Let A be a positive definite $n \times n$ symmetric matrix, and let $x_1, \ldots, x_n$ be n Gaussian random variables, with a probability measure:

$$d\mu(x_1, \ldots, x_n) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \, \mathrm{e}^{-\frac{1}{2}\sum_{i,j} A_{i,j} x_i x_j} \; dx_1 \, dx_2 \, \ldots \, dx_n$$

and let

$$B = A^{-1} \tag{II-2-1}$$

which we call the **propagator**.

Let us denote expectation values with brackets (this is the usual notation in physics):

$$< f(x_1, \ldots, x_n) > \stackrel{\text{def}}{=} \int f(x_1, \ldots, x_n) \; d\mu(x_1, \ldots, x_n).$$

Wick's theorem states that:

Theorem 2.1 (Wick's theorem) *[]*

The expectation value of a product of gaussian random variables, is the sum over all pairings of product of expectation values of pairs.

We have

$$< x_i x_j >= B_{i,j} = \text{propagator}$$

and the expectation value of any odd number of variables is zero, and:

$$< x_{i_1} x_{i_2} \ldots x_{i_{2m}} >= \sum_{\text{pairings}} \prod_{\text{pairs}(k,l)} B_{i_k, i_l}.$$

Example:

$$< x_{i_1} x_{i_2} x_{i_3} x_{i_4} >= B_{i_1,i_2} B_{i_3,i_4} + B_{i_1,i_3} B_{i_2,i_4} + B_{i_1,i_4} B_{i_2,i_3}.$$

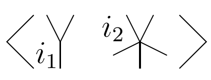
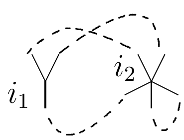
Wick's theorem becomes even more interesting when the indices $i_1, \ldots, i_{2m}$ are not distinct. For instance:

$$< x_{i_1}^2 x_{i_2}^2 >= B_{i_1,i_1} B_{i_2,i_2} + 2 B_{i_1,i_2} B_{i_1,i_2}.$$

Graphs

The best way to write Wick's theorem is diagrammatically. Associate to each pair (i_k, i_l) an edge with weight B_{i_k,i_l}. If an index i_k is repeated, i.e. if it appears as $x_{i_k}^{p_k}$, then we associate to it a vertex with p_k half edges. Wick's theorem says that the expectation value is the sum over all possible ways to link vertices by edges, of the product of propagators corresponding to edges. In other words, draw all possible graphs with the given vertices, and weight each graph by the product of its edge propagators.

Example:

$< x_{i_1}^3 x_{i_2}^5 > =$  $=$  $+ \ \ldots$ 104 other pairings (II-2-2) where this graph has weight

$$B^3_{i_1,i_2}\, B_{i_2,i_2}.$$

In other words, Wick's theorem allows to count the number of ways of gluing vertices (of given valence) by their edges. Such graphs are called **Feynman graphs**. A Feynman graph is a graph, with given vertices, to which we associate a value, wich is the product of the propagators $B_{i,j}$'s of edges:

$$\prod_{e \in \text{edges}} B_{i_e,j_e}.$$

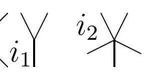
Symmetry factors

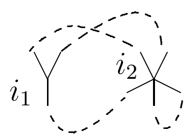
The total number of possible graphs with m edges, is the number of pairings of $2m$ half edges, it is:

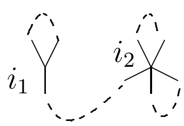
$$(2m-1)!! = (2m-1)(2m-3)(2m-5)\dots 1.$$

However, many of the graphs obtained, are topologically identical, they have the same weight, and it may be more convenient to write only non-topologicaly equivalent graphs, and associate to them an integer factor (the symmetry factor).

For example, the graph displayed in eq.II-2-2, is obtained 60 times, and the only other topological graph is obtained 45 times, which make a total of $60+45=105=7*5*3=7!!$:

$$< x^3_{i_1} x^5_{i_2} > \; = \; \Big\langle$$


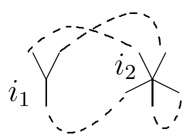
$$\Big\rangle \; = \; 60$$


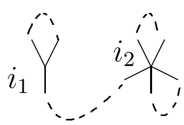
$$+\; 45$$


(II-2-3)

$$= 60\; B^3_{i_1,i_2}\, B_{i_2,i_2} + 45\; B_{i_1,i_2}\, B_{i_1,i_1}\, B^2_{i_2,i_2}.$$

Notice on that example, that both 60 and 45 divide $3! * 5!$:

$$\left\langle \frac{x^3_{i_1}}{3!}\, \frac{x^5_{i_2}}{5!} \right\rangle \; = \; \frac{1}{12}$$


$$+\; \frac{1}{16}$$


(II-2-4)

This is something general: the number of relabelings which leave a graph invariant (i.e. the number of times we obtain the same graph), is equal to the order of the group of relabelings, divided by the number of automorphisms of the graph.

What we call the symmetry factor, is the number of automorphisms of a graph, it appears in the denominator.

To summarize, one may say that **Gaussian expectation values are generating functions for counting (weighted by the inverse of their integer symmetry factor) the number of graphs with given vertices.**

2.2 Matrix gaussian integrals

Let us now apply Wick's theorem, to the computation of gaussian matrix integrals. In that case, the Feynman graphs are going to be fatgraphs also called ribbon-graphs, or maps, or discrete surfaces.

Application of Wick's theorem to matrix integrals

Consider a random hermitean matrix M of size N, with Gaussian probability measure:

$$d\mu_0(M) = \frac{1}{Z_0} \mathrm{e}^{-\frac{N}{2t}\,\mathrm{Tr}\,M^2}\ \prod_{i=1}^{N} dM_{i,i}\ \prod_{i<j} d\mathrm{Re}M_{i,j}\ d\mathrm{Im}M_{i,j}$$

in other words, the variables $M_{i,i}, \mathrm{Re}M_{i,j}, \mathrm{Im}M_{i,j}$ are independent gaussian random variables. Z_0 is the normalization constant such that $\int d\mu_0(M) = 1$:

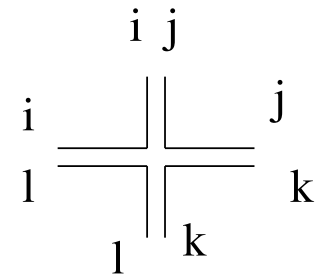
$$Z_0 = 2^N\,(\pi\,t/N)^{\frac{N^2}{2}} \tag{II-2-5}$$

Since $\mathrm{tr}\,M^2 = \sum_{i,j} M_{i,j} M_{j,i}$, the Wick's propagator (defined in eq.II-2-1) is easily computed:

$$< M_{i,j} M_{k,l} >_0 = \frac{t}{N}\,\delta_{i,l}\delta_{j,k}$$

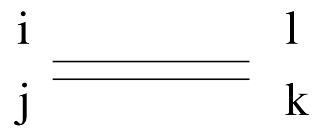
where $<>_0$ means the expectation value with the measure $d\mu_0$.

As a first example, let us compute $<\ \mathrm{Tr}\,M^4\ >_0 = \sum_{i,j,k,l} < M_{i,j} M_{j,k} M_{k,l} M_{l,i} >_0$, which we represent as a vertex with 4 double-line half edges:



We write the half edges as double lines, and associate to each single line its index. Because of the trace, the indices are constant along single lines.

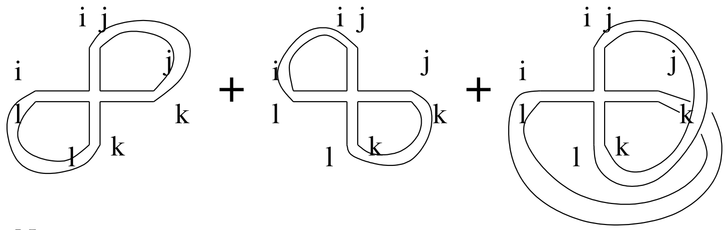
Since the propagator is $< M_{i,j} M_{k,l} >_0 = \frac{t}{N}\,\delta_{i,l}\delta_{j,k}$, it is going to be used to glue together half edges carrying the same oriented pair of indices, we can represent it as an edge:



So, let us compute $<\ \mathrm{Tr}\,M^4\ >_0$:

$$< \frac{N}{4t}\,\mathrm{Tr}\,M^4 >_0$$

$$
\begin{aligned}
&= \frac{N}{4t}\sum_{i,j,k,l} < M_{i,j}M_{j,k}M_{k,l}M_{l,i} >_0 \\
&= \frac{N}{4t}\sum_{i,j,k,l} < M_{i,j}M_{j,k} >_0 < M_{k,l}M_{l,i} >_0 \\
&\qquad + < M_{i,j}M_{l,i} >_0 < M_{j,k}M_{k,l} >_0 + < M_{i,j}M_{k,l} >_0 < M_{j,k}M_{l,i} >_0
\end{aligned}
$$



$$
\begin{aligned}
&= \\
&= \frac{N}{4t}\sum_{i,j,k,l} \frac{t}{N}\delta_{i,k}\delta_{j,j}\ \frac{t}{N}\delta_{k,i}\delta_{l,l} + \frac{t}{N}\delta_{i,i}\delta_{j,l}\ \frac{t}{N}\delta_{j,l}\delta_{k,k} + \frac{t}{N}\delta_{i,l}\delta_{j,k}\ \frac{t}{N}\delta_{j,i}\delta_{k,l} \\
&= \frac{Nt}{4}\Big(\frac{1}{N^2}N^3 + \frac{1}{N^2}N^3 + \frac{1}{N^2}N\Big) \\
&= \frac{t}{4}\Big(N^2 + N^2 + N^0\Big) \\
&= \frac{tN^2}{2} + \frac{tN^0}{4}
\end{aligned}
$$

Notice that there are two steps in that computation:

- the first one consists in applying Wick's theorem, i.e. representing each term as one way of gluing together half edges of the 4-valent vertex with propagators.

- the second step consists in performing the summation over the indices. Notice that the special form of the propagator,with $\delta-$functions of indices, ensures that there is exactly one independent index per single line. The sum over all indices is thus equal to N to the power the number of single lines, i.e. number of faces of the graph.

Since we also have a factor $1/N$ per propagator i.e. per edge, and a factor N in front of the trace, i.e. a factor N per vertex, in the end the total N dependance for a given graph is:

$$N^{\#\text{vertices}-\#\text{edges}+\#\text{faces}} = N^\chi$$

where χ is a topological invariant of the graph, called its **Euler characteristics**.

It should now be clear to the reader that this is something general. The fact that the power of N is a topological invariant, first discovered by 'tHooft [**?**], is the origin of the name "topological expansion".

Wick's theorem ensures that each term in the expectation value corresponds to one way of gluing vertices by their edges, and the sum over indices coming from the traces ensures that the total power of N for each graph is precisely its Euler characteristics, which we summarize as:

$$< \prod_{k=1}^{m} (N\,\mathrm{Tr}\, M^{p_k}) >_0 = \sum_{\mathrm{L-Fat\,Graphs}\,G} N^{\chi(G)}\ t^{\#\text{edges}}$$

where the sum is over the set of (labeled) oriented fat graphs having vertices of valence $p_1,\dots,p_m$ obtained by gluing together half edges.

One should make some remarks:

• the graphs in that sum maybe disconnected

• several graphs may be topologically equivalent in the sum, i.e. if we remove the labelling of indices. The order of the group of relabellings is:

$$\prod_{i=k}^{m} p_k \ \prod_p (\#\{p_i |\ p_i = p\})!$$

indeed, at each vertex of valence p_k one can make p_k rotations of the indices, and if several vertices have the same valence they can be permuted.

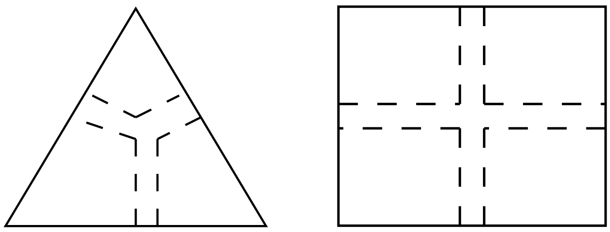
Therefore, it is better to rewrite:

$$\boxed{< \prod_{k=1}^{m} \frac{1}{n_k!} (\frac{N}{k} \operatorname{Tr} M^k)^{n_k} >_0 = \sum_{\text{Fat Graphs}\, G} \frac{1}{\#\text{Aut}(G)} \, N^{\chi(G)} \ \ t^{\#\text{edges}}} \tag{II-2-6}$$

where now the sum is over non-topologicaly equivalent graphs made with n_k k-valent vertices, and $\#\text{Aut}(G)$ is the number of automorphisms of the graph G.

From graphs to maps

Instead of summing over fatgraphs, let us sum over their duals, using the obvious bijection between a graph and its dual. The dual of a k-valent vertex is a k-gon:



gluing together vertices by their half-edges is clearly equivalent to gluing (oriented) polygons together by their sides, and thus we obtain a map. equation eq.II-2-6 can thus be rewritten:

$$< \prod_{k=1}^{m} \frac{1}{n_k!} (\frac{N}{kt} \operatorname{Tr} M^k)^{n_k} >_0 = \sum_{\text{Maps}\, \Sigma} \frac{t^{\#\text{edges}-\#\text{faces}}}{\#\text{Aut}(\Sigma)} \ N^{\chi(\Sigma)}$$

where now the sum is over maps made with n_k k-gons, and $\#\text{Aut}(\Sigma)$ is the number of automorphisms of the map Σ. We have:

• vertices of $G \leftrightarrow$ faces of Σ

• edges of $G \leftrightarrow$ edges of Σ

• faces of $G \leftrightarrow$ vertices of Σ

Notice that the Euler characteristics of a graph and its dual is the same. The Euler-Characteristics is

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{faces}$$

in other words, the power of t is also:

$$t^{\#\text{vertices}-\chi}$$

i.e.

Theorem 2.2

$$\boxed{< \prod_{k=1}^{m} \frac{1}{n_k!}(\frac{N}{kt}\,\mathrm{Tr}\, M^k)^{n_k} >_0 = \sum_{\text{Maps}\,\Sigma} \frac{t^{\#\text{vertices}(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}\,\left(\frac{N}{t}\right)^{\chi(\Sigma)}} \tag{II-2-7}$$

where the sum is over all maps (not necessarily connected) having exactly m faces, with given degrees n_k, $k = 1, \ldots, m$.

3 Generating functions of maps and matrix integrals

3.1 Generating functions for closed maps

Theorem 2.2 implies that the generating function Z_N of eq.I-2-5, which counts non connected maps, is nothing but the formal integral:

Proposition 3.1

$$\begin{aligned} Z_N(t; t_3, t_4, \ldots, t_d) &= \int_{\text{formal}} dM\ \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2t}}\ \mathrm{e}^{\frac{N}{t}\,\mathrm{Tr}\,\left(\frac{t_3}{3}M^3+\frac{t_4}{4}M^4+\ldots\frac{t_d}{d}M^d\right)} \\ &= \sum_{\text{n.c. closed maps}\,\Sigma} \left(\frac{N}{t}\right)^{\chi(\Sigma)}\ t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)}\ \ldots t_d^{n_d(\Sigma)}\ \frac{t^{\#\text{vertices}(\Sigma)}}{\#\mathrm{Aut}(\Sigma)} \end{aligned}$$

where again, formal integral means that we Taylor expand the exponentials of all non quadratic terms, and exchange the Taylor series and the integration. In other words, we perform a formal small t (or also $t_3, t_4, \ldots, t_d$) asymptotic expansion, and order by order we get the number of corresponding maps. The coefficient of t^j is the finite sum of (n.c. = non-connected) closed maps such that $\frac{1}{2}\sum_i (i-2)n_i = j = \#\text{vertices} - \chi$.

Connected maps

When we have a formal generating series counting disconnected objects multiplicatively, it is well known that the logarithm is the generating function which counts only the connected objects, i.e. it is the generating function of eq.I-2-5:

$$\begin{aligned} & \ln\left(Z_N(t; t_3, t_4, \ldots, n_d)\right) \\ = \ & F(t; t_3, t_4, \ldots, n_d; N) \end{aligned}$$

$$= \sum_{\text{closed connected maps } \Sigma} \left(\frac{N}{t}\right)^{2-2g(\Sigma)} \quad t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \, \frac{t^{\#\text{vertices}}}{\#\text{Aut}(\Sigma)}.$$

Again, the coefficient of t^j is the finite sum of connected closed maps such that $\frac{1}{2}\sum_i (i-2)n_i = j = \#\text{vertices}-\chi$. And the Euler characteristics of a connected map is $\chi = 2-2g$ where g is the genus.

Topological expansion: maps of given genus

We thus see, that order by order in the small t expansion, the coefficients of t^j in $N^{-2}\, F$ are polynomials in N^{-2}, and thus we can define generating series of coefficients of a given power of N^{-2g}, we define:

$$F = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} \quad F_g$$

where again we emphasize that this is an equality of formal series in powers of t, and order by order, the sum over g is finite, and the coefficients are polynomials in N^{-2}. F_g is obtained by collecting the coefficients of N^{-2g}, and its computation does not involve any large N limit.

We recognize the generating function of connected closed maps of genus g, of def.I-2-4:

$$F_g(t; t_3, t_4, \dots, n_d) = \sum_v t^v \sum_{\Sigma \in \mathbb{M}_0^{(g)}(v)} \quad t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \, \frac{1}{\#\text{Aut}(\Sigma)}.$$

4 Maps with boundaries or marked faces

4.1 One boundary

So far, we have seen how formal matrix integrals, thanks to Wick's theorem, are counting closed maps were all polygons played similar roles. Now let us count maps with some marked faces.

Consider the following formal matrix integral:

$$<\, \text{Tr}\, M^l > = \frac{\int_{\text{formal}} dM \;\; \text{Tr}\, M^l \;\; \mathrm{e}^{-N\,\text{Tr}\,\frac{M^2}{2t}} \;\; \mathrm{e}^{\frac{N}{t}\,\text{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \dots \frac{t_d}{d}M^d\right)}}{\int_{\text{formal}} dM \;\; \mathrm{e}^{-N\,\text{Tr}\,\frac{M^2}{2t}} \;\; \mathrm{e}^{\frac{N}{t}\,\text{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \dots \frac{t_d}{d}M^d\right)}} \qquad \text{(II-4-1)}$$

The bracket $< . >$ now denotes expectation value with respect to the formal measure $\frac{1}{Z}\,\mathrm{e}^{-N\,\text{Tr}\,\frac{M^2}{2t}} \;\; \mathrm{e}^{\frac{N}{t}\,\text{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \dots \frac{t_d}{d}M^d\right)}\, dM$, whereas in the previous section $< . >_0$ meant the expectation value with respect to the gaussian measure $\frac{1}{Z_0}\,\mathrm{e}^{-N\,\text{Tr}\,\frac{M^2}{2t}}\, dM$.

The numerator in eq.II-4-1 is

$$\int_{\text{formal}} dM \;\; \text{Tr}\, M^l \;\; \mathrm{e}^{-N\,\text{Tr}\,\frac{M^2}{2t}} \;\; \mathrm{e}^{\frac{N}{t}\,\text{Tr}\,\left(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \dots \frac{t_d}{d}M^d\right)}$$

$$
\begin{aligned}
= & \sum_{n_3,\dots,n_d} \frac{N^{n_3}\, t_3^{n_3}}{3^{n_3}\, n_3!} \frac{N^{n_4}\, t_4^{n_4}}{4^{n_4}\, n_4!} \dots \frac{N^{n_d}\, t_d^{n_d}}{d^{n_d}\, n_d!} \;\; \frac{1}{t^{\sum n_j}} \\
& \int \mathrm{Tr}\, M^l \,(\,\mathrm{Tr}\, M^3)^{n_3}\,(\,\mathrm{Tr}\, M^4)^{n_4} \,\dots\,(\,\mathrm{Tr}\, M^d)^{n_d}\;\; \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2t}}
\end{aligned}
$$

it can be computed using Wick's theorem, and it gives a sum over all fatgraphs (or maps) with n_3 triangles, n_4 squares, …, n_d d-gons, and one **marked l-gon**. The sum may include non connected maps, and the role of the denominator in eq.(II-4-1) is precisely to kill all non-connected maps (see section 2.5 in chapter.I).

There should be a symmetry factor $1/\#\mathrm{Aut}(\Sigma)$ counting automorphisms which preserve the marked face, and since there is no factor $\frac{1}{l}$ in front of $\mathrm{Tr}\, M^l$, we get l times the number of maps with no marked edge on the marked face, i.e. we get the number of maps with one marked edge on the marked face. Since there is no N accompanying the $\mathrm{Tr}\, M^l$, the power of N is $\chi - 1 = 2 - 2g - 1$ which is the Euler characteristic of a surface with one boundary. Therefore we recognize the generating function $\mathcal{T}_l$ of eq.(I-2-2) in chapter.I:

$$
\begin{aligned}
<\,\mathrm{Tr}\, M^l\,> \;&=\; \mathcal{T}_l \\
&=\; \sum_{\text{maps }\Sigma\text{ with 1 boundary of length } l} \left(\frac{N}{t}\right)^{1-2g} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}\, \frac{t^{\#\text{vertices}}}{\#\mathrm{Aut}(\Sigma)} \\
&=\; -\operatorname*{Res}_{x\to\infty} x^l\, W_1(x)\, dx.
\end{aligned}
$$

4.2 several boundaries

The previous subsection can be immediately generalized to:

$$
\begin{aligned}
& <\,\mathrm{Tr}\, M^{l_1}\;\mathrm{Tr}\, M^{l_2}\,\dots\,\mathrm{Tr}\, M^{l_k}\,> \\
=\; & \frac{1}{Z}\int_{\text{formal}} dM\;\;\mathrm{Tr}\, M^{l_1}\;\mathrm{Tr}\, M^{l_2}\,\dots\,\mathrm{Tr}\, M^{l_k}\;\; \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^2}{2}}\;\; \mathrm{e}^{N\,\mathrm{Tr}\,\left(\frac{t_3}{3}M^3+\frac{t_4}{4}M^4+\dots\frac{t_d}{d}M^d\right)} \\
=\; & \frac{1}{Z}\,\mathcal{T}^*_{l_1,\dots,l_k} \qquad \text{(II-4-2)}
\end{aligned}
$$

where $\mathcal{T}^*_{l_1,\dots,l_k}$ is the generating function of not necessarily connected maps with k boundaries of lengths $l_1,\dots,l_k$ of all genus.

One obtains connected maps by computing cumulants (see section 2.5 of chapter I), for instance:

$$
<\,\mathrm{Tr}\, M^{l_1}\;\mathrm{Tr}\, M^{l_2}\,>_c = <\,\mathrm{Tr}\, M^{l_1}\;\mathrm{Tr}\, M^{l_2}\,> - <\,\mathrm{Tr}\, M^{l_1}\,><\,\mathrm{Tr}\, M^{l_2}\,>
$$

And thus the cumulants compute connected maps with k boundaries of lengths $l_1,\dots,l_k$:

$$
\begin{aligned}
& \mathcal{T}_{l_1,\dots,l_k} \\
=\; & <\,\mathrm{Tr}\, M^{l_1}\;\mathrm{Tr}\, M^{l_2}\,\dots\,\mathrm{Tr}\, M^{l_k}\,>_c \\
=\; & \sum_{\Sigma\text{ with } k \text{ boundaries of length } l_1,\dots,l_k} \left(\frac{N}{t}\right)^{2-2g-k} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}\, \frac{t^{\#\text{vertices}}}{\#\mathrm{Aut}(\Sigma)}.
\end{aligned}
$$

4.3 Topological expansion for bounded maps of given genus

The Euler characteristics of a connected surface of genus g with k boundaries is:

$$\chi = 2 - 2g - k.$$

Therefore we have:

$$\boxed{<\ \mathrm{Tr}\ M^{l_1}\ \mathrm{Tr}\ M^{l_2}\ \ldots\ \mathrm{Tr}\ M^{l_k}\ >_c = \sum_{g=0}^{\infty} \mathcal{T}^{(g)}_{l_1,\ldots,l_k}\ \left(\frac{N}{t}\right)^{2-2g-k}} \tag{II-4-3}$$

where $\mathcal{T}^{(g)}_{l_1,\ldots,l_k}$ is the generating function defined in chapter I, eq.(I-2-2), which counts connected maps of genus g, with k boundaries of lengths $l_1,\ldots,l_k$.

Once more we emphasize that this equality holds term by term in the powers of t, and for each power, the sum over g is finite, i.e. both left hand side and right hand side are Laurent polynomials in N.

In other words, eq. II-4-3 is not a large N expansion, it is a small t expansion.

4.4 Resolvents

We define the **resolvent**:

$$W_1(x) = \sum_{l=0}^{\infty} \frac{1}{x^{l+1}}\ \mathcal{T}_l = \sum_{l=1}^{\infty} \left\langle \mathrm{Tr}\ \frac{M^l}{x^{l+1}} \right\rangle$$

and conversely:

$$\mathcal{T}_l = -\operatorname*{Res}_{x\to\infty}\ x^l\, W_1(x)\, dx$$

Very often (in particular in physicist's literature), the resolvent is written:

$$W_1(x) =<\ \mathrm{Tr}\ \frac{1}{x-M}\ >$$

which holds in the formal sense, i.e. to each given power of t, the sum over l is finite and each coefficient in the small t or t_j's expansion is a polynomial in $1/x$.

More generally:

$$\begin{aligned} W_k(x_1,\ldots,x_k) &= \sum_{l_1,\ldots,l_k=0}^{\infty} \frac{1}{x_1^{l_1+1}\ldots x_k^{l_k+1}}\ \mathcal{T}_{l_1,\ldots,l_k} \\ &= \sum_{l_1,\ldots,l_k=0}^{\infty} \left\langle \mathrm{Tr}\ \frac{M^{l_1}}{x_1^{l_1+1}}\ \ldots\ \mathrm{Tr}\ \frac{M^{l_k}}{x_k^{l_k+1}} \right\rangle_c \\ &= \left\langle \mathrm{Tr}\,\frac{1}{x_1-M}\ldots\ \mathrm{Tr}\,\frac{1}{x_k-M} \right\rangle_c \\ &= \sum_g \left(\frac{N}{t}\right)^{2-2g-k}\ W_k^{(g)}(x_1,\ldots,x_k) \end{aligned} \tag{II-4-4}$$

The $W_k^{(g)}$ are the same as those of definition 2.2 in chapter I.

5 Loop equations

In this section, we derive a matrix-model proof of Tutte's equation of chapter I. In the matrix model framework, those equations are called "**loop equations**" [?].

Loop equations merely arise from the fact that an integral is invariant under a change of variable, or alternatively from integration by parts. They are sometimes called Schwinger–Dyson equations.

Although loop equations are equivalent to Tutte's equations, it is often easier to integrate by parts in a matrix integral, than finding bijections between sets of maps, and it is much faster to derive loop equations from matrix models than from combinatorics.

Consider the following polynomial expectation value of degree $l = l_1 + \ldots + l_k$:

$$< G^*(M) >= \frac{\int dM\, G^*(M)\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}}{\int dM\ \ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}} \qquad , \qquad G^*(M) = \prod_{j=1}^{k} \operatorname{Tr} M^{l_j}$$

where $\int$ means either the convergent or the formal matrix integral (i.e., to any order in t, a finite sum of convergent gaussian integrals).

We shall derive a recursion relation on the degree $l = (l_1, \ldots, l_k)$.

The method is called loop equations, and it is nothing but integration by parts. It is based on the observation that the integral of a total derivative vanishes, and thus, if $G(M)$ is any matrix valued polynomial function of M (for instance $G(M) = M^{l_1}\ \prod_{j=2}^{k} \operatorname{Tr} M^{l_j}$), we have:

$$\begin{aligned} 0 \;=\; & \sum_{i<j} \int dM\ \frac{\partial}{\partial \mathrm{Re} M_{i,j}}\ \left((G(M))_{ij}\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}\right) \\ & -i \sum_{i<j} \int dM\ \frac{\partial}{\partial \mathrm{Im} M_{i,j}}\ \left((G(M))_{ij}\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}\right) \\ & +\sum_{i=1}^{N} \int dM\ \frac{\partial}{\partial M_{i,i}}\ \left((G(M))_{ii}\ \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}\right) \end{aligned} \qquad \text{(II-5-1)}$$

Choosing

$$G(M) = M^{l_1}\ \prod_{j=2}^{k} \operatorname{Tr} M^{l_j}$$

and after computing the derivatives we get:

$$\begin{aligned} & \sum_{j=0}^{l_1-1} < \operatorname{Tr} M^j \operatorname{Tr} M^{l_1-1-j} \prod_{i=2}^{k} \operatorname{Tr} M^{l_i} > + \sum_{j=2}^{k} l_j\ < \operatorname{Tr} M^{l_j+l_1-1} \prod_{i=2, i\neq j}^{k} \operatorname{Tr} M^{l_i} > \\ = \;& \frac{N}{t} < \operatorname{Tr} \left(M^{l_1} V'(M)\right) \prod_{i=2}^{k} \operatorname{Tr} M^{l_i} > \end{aligned} \qquad \text{(II-5-2)}$$

Again, we emphasize that this equation is valid for both convergent matrix integrals and formal matrix integrals, indeed it is valid for gaussian integrals, and thus for any finite linear combination of gaussian integrals, i.e. formal integrals. In case of formal integrals, those equations are valid, of course, only order by order in t. In other words the loop equations are independent of the order of the integral and the Taylor series expansion.

Using the notations of eq.(II-4-3), we may rewrite the loop equation eq.(II-5-2):

Theorem 5.1 *Loop equations* $\forall g$*:*

$$\boxed{\begin{array}{rl} & \sum_{j=0}^{l_1-1}\left[\sum_{h=0}^{g}\sum_{J\subset L}\mathcal{T}_{j,J}^{(h)}\mathcal{T}_{l_1-1-j,L/J}^{(g-h)}+\mathcal{T}_{j,l_1-1-j,L}^{(g-1)}\right]+\sum_{j=2}^{k}l_j\,\mathcal{T}_{l_j+l_1-1,L/\{j\}}^{(g)} \\ = & \mathcal{T}_{l_1+1,L}^{(g)}-\sum_{j=3}^{d}t_j\mathcal{T}_{l_1+j-1,L}^{(g)} \end{array}} \tag{II-5-3}$$

where we denote collectively $L=\{l_2,\dots,l_k\}$

We recall that $\mathcal{T}_{l_1,l_2,\dots,l_k}^{(g)}$ is the generating function which counts the number of connected maps of genus g with k boundaries of perimeters $l_1,\dots,l_k$, and therefore we have re–derived the generalized **Tutte equation** eq.(I-3-2) of chapter I.

It is interesting to rewrite the loop equations of eq.(II-5-3) in terms of resolvents $W_k^{(g)}$'s defined in eq.(II-4-4). We merely multiply eq.(II-5-3) by $\prod_{i=1}^{k}1/x_i^{l_i+1}$ and sum over $l_1,\dots,l_k$ (to any given power of t, the sum is finite).

Theorem 5.2 *Loop equations. For any k and g, and $L=\{x_2,\dots,x_k\}$, we have:*

$$\begin{aligned} & \sum_{h=0}^{g}\sum_{J\subset L}W_{1+\#J}^{(h)}(x_1,J)W_{k-\#J}^{(g-h)}(x_1,L\setminus J)+W_{k+1}^{(g-1)}(x_1,x_1,L) \\ & +\sum_{j=2}^{k}\frac{\partial}{\partial x_j}\,\frac{W_{k-1}^{(g)}(x_1,L\subset\{x_j\})-W_{k-1}^{(g)}(L)}{x_1-x_j} \\ = \; & V'(x_1)W_k^{(g)}(x_1,L)-P_k^{(g)}(x_1,L) \end{aligned} \tag{II-5-4}$$

where $P_k^{(g)}(x_1,L)$ is a polynomial in x_1, of degree $d-3$ (except $P_1^{(0)}$ which is of degree $d-2$):

$$P_k^{(g)}(x_1,x_2,\dots,x_k)=-\sum_{j=2}^{d-1}t_{j+1}\sum_{i=0}^{j-1}x_1^i\sum_{l_2,\dots,l_k=1}^{\infty}\frac{\mathcal{T}_{j-1-i,l_2,\dots,l_k}^{(g)}}{x_2^{l_2+1}\dots x_k^{l_k+1}}+t\,\delta_{g,0}\delta_{k,1}$$

proof:
Indeed, if we expand both sides of eq.(II-5-4) in powers of $x_1\to\infty$, and indentify the coefficients on both side, we find that the negative powers of the x_i's give precisely the loop equations eq.(II-5-3), whereas the coefficients of positive powers of x_1 cancel due to the definition of $P_k^{(g)}$, which is exactely the positive part of $V'(x_1)W_k^{(g)}$:

$$P_k^{(g)}(x_1,x_2,\dots,x_k)=\mathop{\mathrm{Pol}}_{x_1\to\infty}\left(V'(x_1)\,W_k^{(g)}(x_1,x_2,\dots,x_k)\right)$$

where Pol means that we keep only the polynomial part, i.e. the positive part of the Laurent series at $x_1 \to \infty$. $\Box$

6 Loop equations and Virasoro constraints

We have seen two derivations of the loop equations. One combinatoric proof in chapter I, based on Tutte's method, corresponding to recursively removing a marked edge, and one proof based on integration by parts in the formal matrix integral in chapter II. However, there exist other possible derivations.

In particular, in string theory and quantum gravity, it is known that partition functions must satisfy Virasoro constraints. Here, we show how to rewrite the loop equations for generating functions of maps, as Virasoro constraints.

We write the potential:

$$V(x) = -\sum_{j=1}^{\infty} \frac{t_j}{j}\, x^j$$

In the end, we will be interested in $t_1 = 0, t_2 = -1$ and $t_j = 0$ if $j > d$.

It is easy to see from the definitions of our generating functions, and particularly on the formal matrix integral, that:

$$\mathcal{T}_j^{(g)} = j\,\frac{\partial F_g}{\partial t_j} \qquad , \qquad \mathcal{T}_{j_1,j_2}^{(g)} = j_1\, j_2\, \frac{\partial^2 F_g}{\partial t_{j_1} \partial t_{j_2}}$$

and therefore the loop equations eq.(II-5-3) for $k = 1$ can be rewritten:

$$\forall k \geq -1 \qquad , \qquad \mathcal{V}_k . Z = 0 \tag{II-6-1}$$

where we have defined the operator:

$$\mathcal{V}_k = \frac{1}{N^2} \sum_{j=1}^{k-1} j\,(k-j)\, \frac{\partial}{\partial t_j}\, \frac{\partial}{\partial t_{k-j}} + \sum_{j=2}^{d} (k+j)\, t_j\, \frac{\partial}{\partial t_{k+j}}.$$

The differential operators $\mathcal{V}_k$ form a representation of (the positive part of) the Virasoro algebra, indeed one easily verifies that they satisfy:

$$[\mathcal{V}_k, \mathcal{V}_j] = (k-j)\mathcal{V}_{k+j}$$

This method has been extensively used by physicists, but we shall not pursue in that direction in this book.

An important property, is that eq.(II-6-1) is a linear equation for Z, and thus, linear combinations of solutions are also solutions.

7 Summary Maps and matrix integrals

Let us summarize the concepts introduced in this chapter:

- Formal integral

$$Z_N = \int_{\text{formal}} \mathrm{e}^{-\frac{N}{t}\operatorname{Tr} V(M)}\, dM \qquad , \qquad V(M) = \frac{M^2}{2} - \sum_{k=3}^{d} \frac{t_k}{k} M^k$$

  where $\int_{\text{formal}}$ means that we exchange the order of the integral and the Taylor expansion of the exponentials of the t_k's.

- $\mathbb{M}_0^{(g)}(v)$ = finite set of **connected** maps of **genus** g and no boundary, with v vertices, obtained by gluing n_3 triangles, n_4 squares, n_5 pentagons,..., n_d $d-$gons.

  Generating function:

$$\begin{aligned}\ln Z_N &= \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} F_g \\ &= \sum_{j=0}^{\infty} t^j \sum_{v+2g-2=j} N^{2-2g} \sum_{\Sigma\in\mathbb{M}_0^{(g)}(v)} \frac{t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}\end{aligned}$$

  We also denote:

$$F_g = W_0^{(g)}.$$

- $\mathbb{M}_k^{(g)}(v)$ = connected **maps of genus** g with v vertices, obtained by gluing n_3 triangles, n_4 squares, n_5 pentagons, and k boundaries of length $l_1, \dots, l_k$.

  Generating function:

$$\begin{aligned}&< \operatorname{Tr} M^{l_1} \operatorname{Tr} M^{l_2} \dots \operatorname{Tr} M^{l_k} >_c \\ =& \sum_{j=0}^{\infty} t^j \sum_{v+2g+k-2=j} N^{2-2g-k} \sum_{\Sigma\in\mathbb{M}_k^{(g)}(v), \delta\Sigma=\{l_1,\dots,l_k\}} \frac{t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}}{\#\mathrm{Aut}(\Sigma)} \\ =& \sum_{g} \left(\frac{N}{t}\right)^{2-2g-k} \mathcal{T}^{(g)}_{l_1,\dots,l_k}.\end{aligned}$$

- Resolvents for connected **maps of genus** g and with k boundaries.

  Generating function:

$$\begin{aligned}&W_k(x_1,\dots,x_k) \\ =& \sum_g \left(\frac{N}{t}\right)^{2-2g-k} W_k^{(g)}(x_1,\dots,x_k) \\ =& < \operatorname{Tr} \frac{1}{x_1 - M} \dots \operatorname{Tr} \frac{1}{x_k - M} >_c \\ =& \sum_{j=0}^{\infty} t^j \sum_{v+2g+k-2=j} N^{2-2g-k} \sum_{\Sigma\in\mathbb{M}_{g,k}(v)} \frac{t_3^{n_3} t_4^{n_4} \dots t_d^{n_d}}{x_1^{l_1+1} \dots x_k^{l_k+1}} \frac{1}{\#\mathrm{Aut}(\Sigma)}\ .\end{aligned}$$

- Loop equations (Tutte's equations):

$$\sum_{j=0}^{l_1-1}\Big[\sum_{h=0}^{g}\sum_{J\subset L}\mathcal{T}^{(h)}_{j,J}\mathcal{T}^{(g-h)}_{l_1-1-j,L/J}+\mathcal{T}^{(g-1)}_{j,l_1-1-j,L}\Big]+\sum_{j=2}^{k} l_j\,\mathcal{T}^{(g)}_{l_j+l_1-1,L/\{j\}}$$
$$= \mathcal{T}^{(g)}_{l_1+1,L}-\sum_{j=3}^{d} t_j\mathcal{T}^{(g)}_{l_1+j-1,L}$$

where $L=\{l_2,\dots,l_k\}$. Equivalently, the loop equations can be written in terms of $W_k^{(g)}$'s and with $L=\{x_2,\dots,x_k\}$:

$$\sum_{h=0}^{g}\sum_{J\subset L}W^{(h)}_{1+\#J}(x_1,J)W^{(g-h)}_{k-\#J}(x_1,L\setminus J)+W^{(g-1)}_{k+1}(x_1,x_1,L)$$
$$+\sum_{j=2}^{k}\frac{\partial}{\partial x_j}\,\frac{W^{(g)}_{k-1}(x_1,L\setminus\{x_j\})-W^{(g)}_{k-1}(L)}{x_1-x_j}$$
$$= V'(x_1)W_k^{(g)}(x_1,L)-P_k^{(g)}(x_1,L)$$

where $L=\{x_2,\dots,x_k\}$, and $P_k^{(g)}(x_1,L)=\mathrm{Pol}_{x_1}\,V'(x_1)\,W_k^{(g)}(x_1,L)$ is a polynomial in the variable x_1, of degree $d-3$, except $P_1^{(0)}$ which is of degree $d-2$.

8 Exercises

Exercise 1:

For the quartic formal matrix integral

$$Z=\frac{1}{Z_0}\int_{\text{formal}} dM\,\mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\,\frac{M^2}{2}-\frac{t_4 M^4}{4}}=1+\frac{Nt_4}{4t}\left\langle \mathrm{tr}\,M^4\right\rangle+\frac{1}{2}\left(\frac{Nt_4}{4t}\right)^2\left\langle(\mathrm{tr}\,M^4)^2\right\rangle+O(t_4^3)$$

using Wick's theorem, recover the generating function of quadrangulations

$$\ln Z=F=\frac{t_4}{4}\,(2N^2+1)+\frac{t_4^2}{8}\,(9N^2+15)+O(t_4^3).$$

Exercise 2: Prove that with any potential:

$$<\,\mathrm{Tr}\,V'(M)>=0\qquad,\qquad\frac{t}{N}\,<\,\mathrm{Tr}\,MV'(M)>=t^2$$

Hint: this is a loop equation, use integration by parts.

Exercise 3: Prove that for quadrangulations (i.e. with $V(M)=\frac{M^2}{2}-t_4\,\frac{M^4}{4}$):

$$\frac{\partial F}{\partial t}=\frac{Nt_4}{4t^2}\,\mathcal{T}_4$$

answer: hint: use exercise 2, and don't forget the t dependance of the normalization factor Z_0 in eq.(II-2-5).