Chapter I

Maps and discrete surfaces

In this chapter we define maps, which are discrete surfaces obtained by gluing polygons along their sides, and we define generating functions to count them. We also derive Tutte's equations, which are recursive equations satisfied by those generating functions.

We will also rederive Tutte's equations in the matrix model language in chapter.II, and we will give their solution for any topology in chapter.III.

Several classical books exist about maps, for instance: C. Berge [**?**], W.T. Tutte [**?**, **?**], J.L. Gross, T.W. Tucker [**?**], as well as [**?**, **?**, **?**, **?**].

1 Gluing polygons

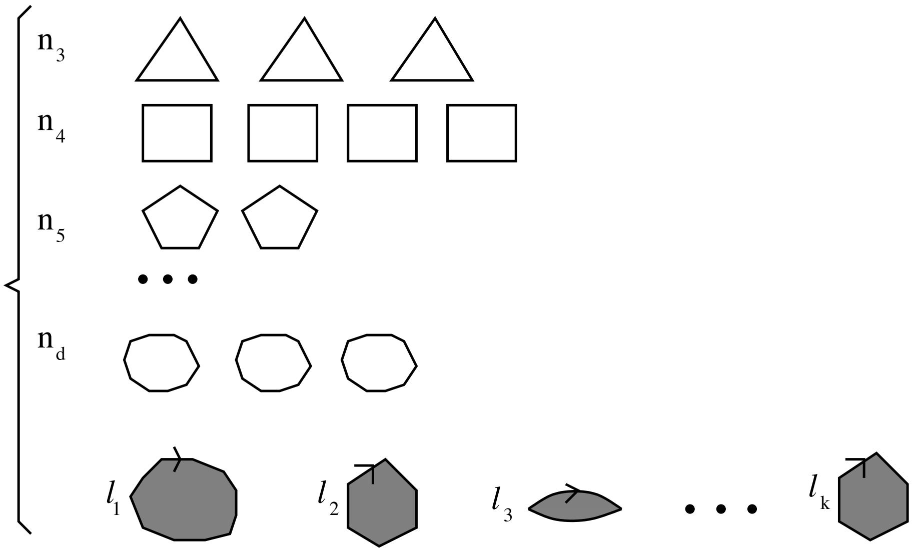
The idea of a map, is a collection of countries and seas on a world. However, in our case, the world is not a sphere or a plane, but a surface with an almost arbitrary topology, and the countries are polygons. Maps are also called **"discrete surfaces"**.

From now on, we shall consider only **orientable surfaces**, the case of maps on non-orientable surfaces can be treated in a similar approach, and satisfies the same topological recursion, but this subject is under development at the time this book is being written.

1.1 Intuitive definition

Here, we give an intuitive and informal definition of a map. A formal definition is given in the next subsection, or can be found in [**?**, **?**, **?**, **?**, **?**, **?**], and examples can be found in [**?**]. The idea is a collection of polygons glued together side by side. We consider two kinds of polygons: some unmarked ones, and some marked ones.

Definition 1.1 (Intuitive definition) *A map, with n_3 unmarked triangles, n_4 unmarked quadrangles, n_5 unmarked pentagons, ..., n_d unmarked $d-$angles, and with k labeled "boundaries" of lengths $l_1, \dots, l_k$ (a boundary of length l is a marked polygon with l sides, and with one marked clockwise oriented side, boundaries are labeled from 1 to k), is a connected gluing of those polygonal pieces along their edges:*



or more precisely, it is the equivalence class of such gluings under graph isomorphisms (i.e. composition of permutations and rotations of the unmarked $i-$gons, which preserve the oriented marked edges).

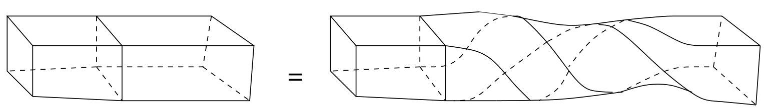
Moreover, we require that unmarked polygons have at least 3 sides, and marked polygons have length at least $l_i \geq 1$.

Remark 1.1 As mentioned above, this is only an intuitive definition. The actual definition is given in def 1.2 below, or in [**?**, **?**, **?**, **?**, **?**, **?**].

Remark 1.2 Notice that nothing in the definition prevents from gluing a side of a polygon to another side of the same polygon, in particular to an adjacent side. This means that the corresponding surfaces can be rather singular.

Remark 1.3 We recall that we consider **oriented polygons**, thus the two faces of a polygon can be imagined with 2 different colors, and polygons must be glued by their sides together, respecting the color. The boundaries have a marked edge on their side, and by convention, we represent the marked edge with an arrow, in such a way that the boundary's face sits on the right side of the marked edge.

Remark 1.4 A "gluing" of polygons means a set of incidence relations, i.e. which edge is glued to which edge. For example, twists are irrelevant.



Remark 1.5 Let us emphasize again that we assume that each unmarked polygonal face (which is not a boundary) has a degree ≥ 3, and $\leq d$:

$$3 \leq \text{degree of unmarked faces} \leq d$$

whereas for the boundaries we only require that:

$$l_i \geq 1$$

The union of all faces of the map, is a surface, and the map can be seen as an embedding of a graph into a surface. We may consider a marked point at the center of each boundary face, and thus a map with k boundaries is naturally embedded into a surface with k-marked points.

The intuitive way of thinking about a boundary would be to exclude the boundaries from the surface, and thus a map with k boundaries would be naturally embedded into a surface with k "holes", however one should be careful with that too simple picture. Indeed, one should notice that those surfaces can be rather singular, because nothing in our definition prevents from gluing a polygon, and in particular a boundary, to itself or to another boundary. This means that, although the interior of the boundary is an open disk, the boundary with its border might not be a disk, it might be not simply connected. Therefore, if we remove the interior of boundaries, the remaining surface may be singular, and if we remove the boundaries together with their borders, the removed parts are not necessarily discs removed from the surface.

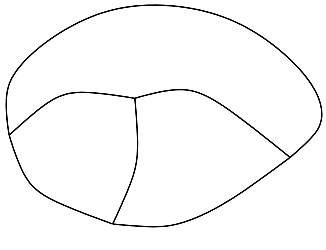
Examples

Maps with no boundary ($k = 0$) are called "closed maps", and maps with boundaries are called "open maps" or "bordered maps".

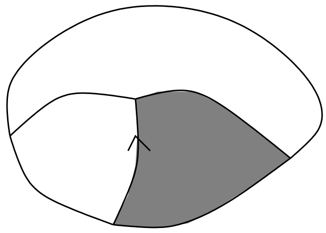
Maps with $k = 1$ boundary, i.e. with only one marked oriented edge, are also called **rooted maps**, and we shall see below that they play an important role.

A map made of only triangles is called a triangulation, a map made with quadrangles is a quadrangulation. A map with only even polygons is called "bipartite".

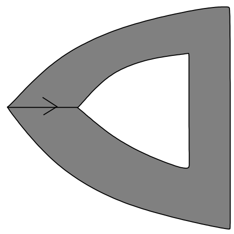
Example of a triangulation with $n_3 = 4$ triangles (the one on the exterior has 3 sides as well), drawn on the plane, i.e. on the Riemann sphere:



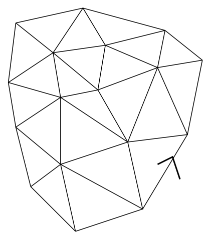
Example of a triangulation with $n_3 = 3$ triangles (one is the exterior one), and one boundary $k = 1$ of length $l_1 = 3$, drawn on the plane, i.e. on the Riemann sphere:



Example of a map with $n_3 = 2$ triangles, and with $k = 1$ boundary of length $l_1 = 8$. Notice that the octogon is glued with itself along the marked edge.



Example of a planar map with $n_3 = 22$ triangles, and one boundary (the exterior) of length $l_1 = 10$:

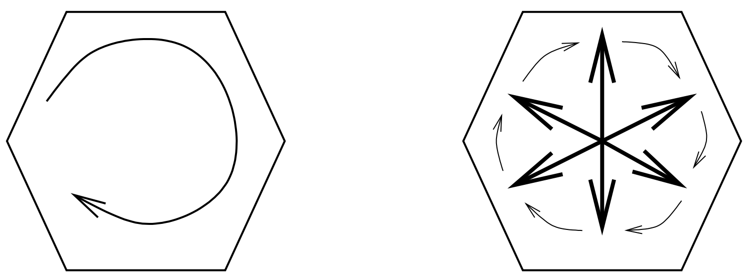


1.2 Formal definition

There exists several equivalent definitions of maps. Let us give the following ones, and refer the reader to the literature [] for other ones (for instance in terms of fatgraphs, or trees,...).

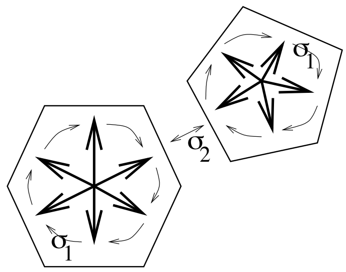
Definition with permutations

A polygon can be seen as a set of edges, together with a cyclic permutation which encodes which edge is next to which edge along the oriented polygon's side. In a dual manner, an edge of the polygon can be seen as an half edge from the center of the polygon to the edge. We call this half–edge a "dart".



Then, gluing edges together means gluing two darts together. The gluing can thus be encoded into an involutive application from the set of darts to itself, and with no fixed point (a dart can't be glued to itself).

Definition 1.2 *A labeled map $G = (B, \sigma_1, \sigma_2)$ is the data of a finite ensemble B (whose elements are called "darts" or "half-edges") of even cardinal, and two permutations σ_1 and σ_2 of elements of B, and such that σ_2 is an involution without fixed points. The cycles of σ_1 are called faces, the cycles of σ_2 are called edges, and the cycles of $\sigma_1 \circ \sigma_2$ are called vertices.*



2 labeled maps (B, σ_1, σ_2) and $(B', \sigma_1', \sigma_2')$ are isomorphic iff there exists a bijection $\phi : B \to B'$, such that $\sigma_1' = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2' = \phi \circ \sigma_2 \circ \phi^{-1}$.

A map is an equivalence class of labeled maps modulo isomorphisms.

The map is said connected if σ_1 and σ_2 act transitively, i.e. if any two elements of B can be related by a sequence of applications of σ_1 and σ_2.

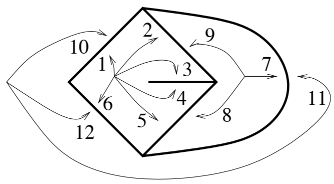
The Euler charcteristics of a map is $\chi = \#\text{faces} - \#\text{edges} + \#\text{vertices}$.

The Automorphism group of a labeled map, is the set of bijections $\phi : B \to B$, such that $\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}$. If two labeled maps are isomorphic, their automorphism groups are isomorphic, and in particular the number of automorphisms of a map is well defined (it depends only on the map, not on a labeled map).

Example: the following two permutations encode a labeled map with 1 hexagon and 2 triangles

σ_1 = (2,3,4,5,6,1,8,9,7,11,12,10)= (2,3,4,5,6,1)(8,9,7)(11,12,10)

σ_2 = (10,9,4,3,8,12,11,5,2,1,7,11)= (10,1)(9,2)(4,3)(8,5)(12,6)(11,7



This definition can be modified in order to have boundaries, i.e. marked faces.

Definition 1.3 *A labeled map $G = (B^*, B, \sigma_1, \sigma_2)$ is the data of 2 finite ensembles B and B^* (whose elements are called "darts" or "half-edges"), such that $B \cup B^*$ has even cardinal, and two permutations σ_1 and σ_2 of elements of $B \cup B^*$, such that σ_2 is an involution without fixed points, and every cycle of σ_1 contains at most one element of B^*. The cycles of σ_1 are called faces, the cycles of σ_2 are called edges, and the cycles* *of $\sigma_1 \circ \sigma_2$ are called vertices. The cycles of σ_1 which contain one element of B^* are called marked faces, and the elements of B^* are marked edges. Each marked face has exactly one marked edge.*

2 labeled maps $(B, B^*, \sigma_1, \sigma_2)$ and $(B', B'^*, \sigma'_1, \sigma'_2)$ are isomorphic iff there exists a bijection $\phi : B \cup B^* \to B' \cup B'^*$, such that $\phi(B^*) = B'^*$ and $\sigma'_1 = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma'_2 = \phi \circ \sigma_2 \circ \phi^{-1}$.

A map is an equivalence class of labeled maps modulo isomorphisms.

The map is said connected if σ_1 and σ_2 act transitively, i.e. if any two elements of $B \cup B^*$ can be related by a sequence of applications of σ_1 and σ_2.

The Euler charcteristics of a map is $\chi = \#\text{faces} - \#\text{edges} + \#\text{vertices} - \#B^*$ (we don't count marked faces).

The Automorphism group of a labeled map, is the set of bijections $\phi : B \cup B^* \to B \cup B^*$, such that the restriction of ϕ to B^* is the identity, and $\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}$ and $\sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}$. The number of automorphisms depends only on the map.

1.3 Topology

The topology of a map, is the topology of the surface with the interior of marked faces removed, i.e. with k discs removed, it is entirely characterized by its Euler characteristics:

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{unmarked faces}.$$

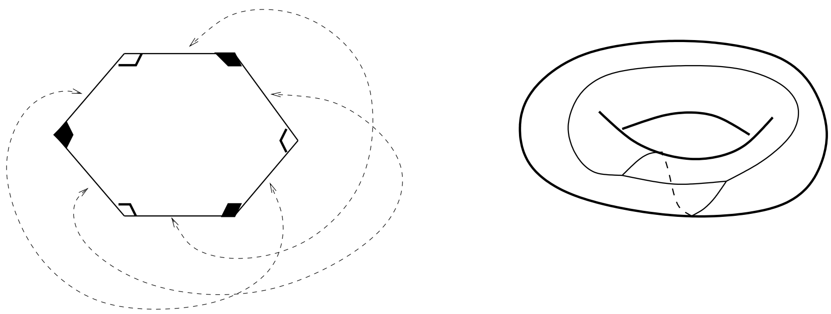
It is worth:

$$\chi = 2 - 2g - k$$

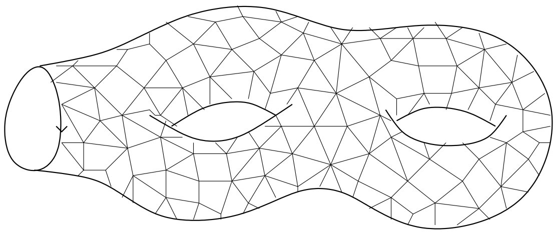
where g is the genus, i.e. the "number of handles", and for non closed surfaces, k is the number of boundaries.

If $g = 0$, i.e. if the surface has the topology of a sphere (with k discs removed), we say that it is a **planar map**.

Example of a map with no boundary ($k = 0$) and only 1 hexagon ($n_6 = 1$), whose opposite sides are glued together. There is 1 face, 3 edges, and 2 vertices (one black and one white in the picture below), i.e. $\chi = 0$, i.e. $g = 1$. This map cannot be drawn on the plane, it can be drawn on a torus:



Other example: the following map has genus $g = 2$, and $k = 1$ boundary, i.e. $\chi = -3$.



1.4 Symmetry factor

An automorphism of a map defined as in def 1.2 by a set of darts $B \cup B^*$ and two permutations σ_1, σ_2, is a bijective map $\phi : B \cup B^* \to B \cup B^*$, such that $\phi|_{B^*} = \mathrm{Id}$ and

$$\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1} \qquad , \qquad \sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}.$$

Since $B \cup B^*$ is a finite set, there can be only a finite number of automorphisms for each map. There is always an obvious automorphism which is the identity, and automorphisms always have a group structure, subgroup of the group of permutations of $B \cup B^*$.

Definition 1.4 *The symmetry factor* $\#\,\mathrm{Aut}$ *of a map is the number of its automorphisms:*

$$\#\,\mathrm{Aut}$$

For generic maps, there is only one automorphism (identity), and $\#\mathrm{Aut} = 1$.

Proposition 1.1 *For open graphs with $k \geq 1$ boundaries, the group of automorphisms is always trivial*

$$k \geq 1 \quad \Rightarrow \quad \#\mathrm{Aut} = 1.$$

proof:

Since B^* is not empty, and since $\phi|_{B^*} = \mathrm{Id}$, there is at least one element for which $\phi(x) = x$. This implies that $\sigma_1(x)$ and $\sigma_2(x)$ are also fixed by ϕ, and by an easy recursion, since the map is connected, i.e. since σ_1 and σ_2 act transitively, every element of B can be linked to x by σ_1 and σ_2 and thus is a fixed point of ϕ. This implies that $\phi = \mathrm{Id}$, and thus the only possible automorphism is the identity. $\square$

There is another way of computing the symmetry factor.

Definition 1.5 *For a given map $m = (B, B^*, \sigma_1, \sigma_2)$, let G_m be the group of relabelings of unmarked darts leaving faces invariants, i.e. the group of bijections $\phi : B \cup B^* \to B \cup B^*$ such that $\phi|_{B^*} = \mathrm{Id}$ and $\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}$*

$$G_m = \{\phi \mid \sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}\}$$

The group G_m acts on the set of permutation of darts by conjugation, and in particular, $\mathrm{Aut}(m)$ *is the stabilizer of σ_2 under the G_m action, i.e. the subgroup of G_m which fixes σ_2:*

$$\mathrm{Aut}(m) = \{\phi \in G_m \mid \sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1}\}.$$

We define the set of "gluings" as the orbit of σ_2 under the G_m action, i.e.

$$\mathrm{Gluings}(m) = \{\phi \circ \sigma_2 \circ \phi^{-1} \mid \phi \in G_m\}$$

The following is a classical result of group theory

Lemma 1.1 (Orbit-stabilizer theorem) *Let G be a group of permutations, acting on set X, and let $x \in X$.*

The orbit of x is $G.x = \{g.x \mid g \in G\}$. The stabilizer of x is $G_x = \{g \in G \mid g.x = x\}$. The quotient G/G_x is the set of equivalence classes of G modulo G_x, i.e. an element of G/G_x can be written $g.G_x = \{g.h \mid h \in G_x\}$.

The orbit-stabilizer theorem says that

$$G.x \sim G/G_x$$

and thus

$$|G| = |G.x|.|G/G_x|.$$

proof:

The bijection $G_x \to G/G_x$ is $g.x \mapsto g.G_x$. This map is well defined, because if there exists g and g' such that $g.x = g'.x$, this means that $g^{-1} \circ g'$ belongs to G_x and thus $g.G_x = g'.G_x$. The inverse map is also well defined for the same reason. □

The "orbit-stabilizer theorem" implies

Proposition 1.2 (Symmetry factor and gluing number) *Let Σ be a map with n_3 triangles, n_4 quadrangles,... n_d d-gons,... We have:*

$$\#\mathrm{Aut} \times \#\mathrm{gluings} = \prod_{j=3}^{d} j^{n_j}\, n_j! \tag{I-1-1}$$

where #gluings is the number of ways of obtaining the map Σ by gluing together the n_3 triangles, n_4 quadrangles,... n_d d-gons, and k marked faces of length $l_1, \dots, l_k$ with marked edges.

proof:

$\mathrm{Gluings}(m)$ is the orbit of σ_2 under G_m, while $\mathrm{Aut}(m)$ is the stabilizer of σ_2, therefore the orbit-stabilizer theorem implies that

$$\#\mathrm{Aut} \times \#\mathrm{gluings} = \#G_m.$$

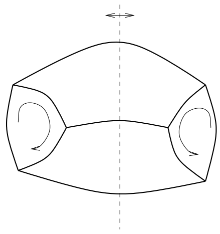
G_m is the conjugacy class of σ_1, it depends only on its cycles, i.e. the faces of the map, and we said there are n_j faces of size j. Each cycle of length j can be conjugated by

j possible cyclic permutations, and cycles of same length can be permuted together, therefore

$$\#G_m = \prod_{j=3}^{d} j^{n_j}\, n_j!$$

$\square$

Example of a closed planar map with no marked edge ($k = 0$, $g = 0$, $\chi = 2$) with 2 triangles and 3 quadrangles (including the exterior one), drawn on the sphere. It has 6 vertices, 9 edges, and 5 faces. Its symmetry factor is 6.



Indeed, Aut $= \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_2$ is generated by the automorphism which exchanges the 2 triangles (central symmetry on the figure), and $\mathbb{Z}_3$ is generated by the simultaneous rotation of the triangles (which permutes cyclically the 3 quadrangles).

How many gluings of 2 triangles and 3 quadrangles correspond to that map ? Chose one of the triangles, and label its 3 sides 1,2,3. There is $6 = 3!$ ways of gluing the 3 quadrangles to its 3 sides, and each quadrangle can be glued to the triangle along any of its 4 edges. Then, there is only 3 possibilities to glue the last triangle. Therefore:

$$\#\text{gluings} = 3! \times 4^3 \times 3 = 2^7 \times 3^2.$$

And thus we verify eq.I-1-1 on that example:

$$6 = \frac{3^2\, 2! \,\times\, 4^3\, 3!}{2^7 \times 3^2}.$$

2 Generating functions for counting maps

Our goal is to count the number of maps having a fixed topology, and fixed numbers of polygons of given sizes, fixed number of boundaries,... Those numbers of maps can be collectively encoded in some generating functions. Let us define them below.

2.1 Maps with fixed number of vertices

Definition 2.1 *Let $\mathbb{M}_k^{(g)}(v)$, be the set of connected maps of genus g, (with umarked faces of degree ≥ 3 and $\leq d$), and k boundaries (marked faces of degree ≥ 1 with one marked edge), and such that the total number of vertices is v. In addition we define $\mathbb{M}_1^{(0)}(1) = \{.\}$ i.e. we have defined a virtual planar rooted map with 1-vertex to be a point, it has no faces, and its unique boundary has length $l_1 = 0$.*

Theorem 2.1 $\mathbb{M}_k^{(g)}(v)$ *is a finite set.*

proof:

Indeed, for any map $\Sigma \in \mathbb{M}_k^{(g)}(v)$, write its Euler characteristics:

$$2-2g = \overbrace{k+\sum_{i=3}^{d} n_i(\Sigma)}^{\#\text{ of faces}} - e(\Sigma) + v$$

where $n_i(\Sigma)$ is the number of faces of degree i, and where $e(\Sigma)$ is the number of edges of Σ, that is, half the number of half-edges:

$$2e(\Sigma) = \sum_{i=1}^{k} l_i(\Sigma) + \sum_{i=3}^{d} i\, n_i(\Sigma)$$

thus we have:

$$v+2g-2+k = \frac{1}{2}\sum_{i=1}^{k} l_i(\Sigma) + \frac{1}{2}\sum_{i=3}^{d}(i-2)n_i(\Sigma) \tag{I-2-1}$$

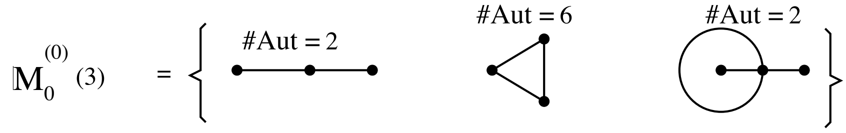
and since $i \geq 3$ in the last sum we have $i-2 \geq 1$ and therefore we find the inequality:

$$v+2g-2+k \geq \frac{1}{2}\sum_{i=1}^{k} l_i(\Sigma) + \frac{1}{2}\sum_{i=3}^{d} n_i(\Sigma)$$

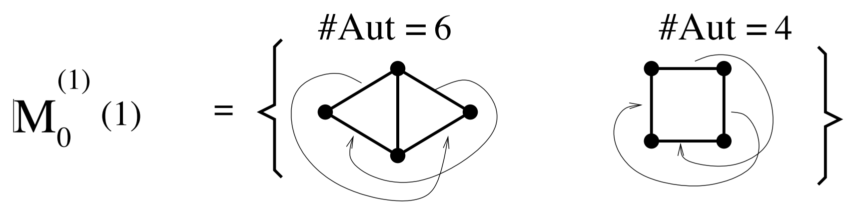
in particular, this inequality implies that n_i and l_i are bounded, and therefore there is a finite number of such maps. $\square$

Examples:

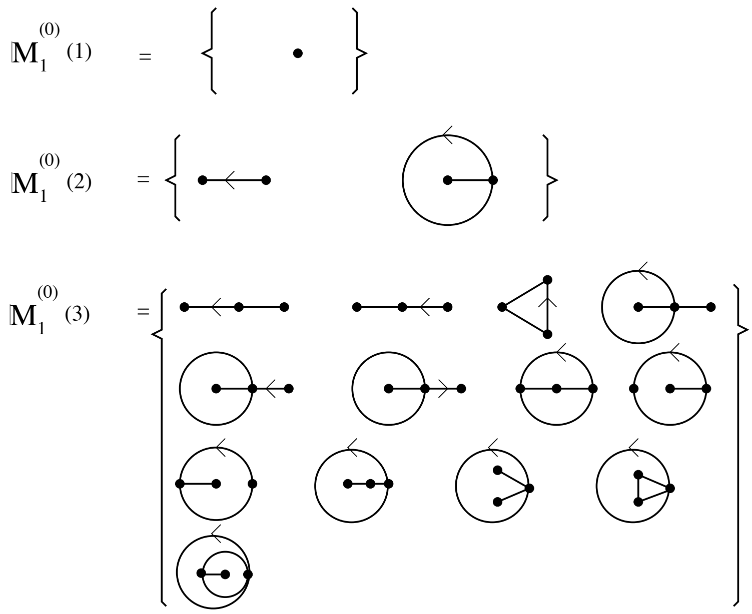
• planar maps with no marked faces and 3 vertices:



• genus 1 maps with no marked faces and 1 vertex



• planar maps with one marked face (the marked face is on the right of the marked edge, i.e. on the exterior):



Definition 2.2 *We define the generating function of maps of genus g with k boundaries as the formal power series in t:*

$$\begin{aligned} & W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d;t) \\ = & \frac{t\,\delta_{k,1}\delta_{g,0}}{x_1} + \sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\mathbb{M}_k^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\,t_4^{n_4(\Sigma)}\,\dots\,t_d^{n_d(\Sigma)}}{x_1^{1+l_1(\Sigma)}\,x_2^{1+l_2(\Sigma)}\,\dots\,x_k^{1+l_k(\Sigma)}}\;\frac{1}{\#\mathrm{Aut}(\Sigma)} \end{aligned}$$

Notational remark: In the rest of this book, we shall write only the dependence in the x_i's explicitly, whereas the dependence in $t, t_3, \dots, t_d$, will be implicitly assumed, by convention we write:

$$W_k^{(g)}(x_1,\dots,x_k;t_3,\dots,t_d;t) \equiv W_k^{(g)}(x_1,\dots,x_k)$$

By convention we also denote:

$$F_g \equiv W_0^{(g)}.$$

Examples:

$$W_1^{(0)} = \frac{t}{x} + t^2\left(\frac{1}{x^3}+\frac{t_3}{x^2}\right) + t^3\left(\frac{2}{x^5}+\frac{4t_3}{x^4}+\frac{t_3^2}{x^3}+\frac{2t_4}{x^3}+\frac{2t_5}{x^2}+\frac{2t_3t_4}{x^2}\right) + O(t^4)$$

$$F_0 = W_0^{(0)} = t^3\,\left(\frac{1}{2}\,t_4 + \left(\frac{1}{6}+\frac{1}{2}\right)\,t_3^2\right) + O(t^4)$$

$$F_1 = W_0^{(1)} = t\,\left(\frac{1}{6}\,t_3^2 + \frac{1}{4}\,t_4\right) + O(t^2)$$

Remark 2.1 As usual in combinatorics, $W_k^{(g)}$ is a generating function, that is a **formal power series** in t, or also called an "asymptotic series". It is meaningful even when it is not convergent. It is nothing but a convenient short hand notation for the collection of all coefficients. However, it turns out (see chapter.III) that each $W_n^{(g)}$ happens to be an algebraic function of t, and therefore it has a finite radius of convergency. The behaviour in the vicinity of the radius of convergency can be used to find the asymptotic numbers of large maps (see chapter V).

Remark 2.2 Notice that for each v, the sum over $\Sigma \in \mathbb{M}_k^{(g)}(v)$ is finite, and therefore the coefficient of t^v in $W_k^{(g)}(x_1, \dots, x_k)$ is a polynomial in the $1/x_i$'s.

2.2 Fixed boundary lengths

If we wish to compute generating functions for maps with fixed boundary lengths l_i, we simply pick the coefficient of $1/x_i^{1+l_i}$ by taking a residue. We define:

Definition 2.3

$$\mathcal{T}_{l_1,\dots,l_k}^{(g)} = (-1)^k \operatorname*{Res}_{x_1\to\infty} \dots \operatorname*{Res}_{x_k\to\infty} x_1^{l_1} \dots x_k^{l_k}\, W_k^{(g)}(x_1, \dots, x_k)\, dx_1 \dots dx_k \tag{I-2-2}$$

Alternatively:

$$\mathcal{T}_{l_1,\dots,l_k}^{(g)} = t\, \delta_{k,1}\delta_{g,0}\delta_{l_1,0} + \sum_{v=1}^{\infty} t^v \sum_{\Sigma\in\mathbb{M}_k^{(g)}(v)} \frac{t_3^{n_3(\Sigma)}\, t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)}}{\#\mathrm{Aut}(\Sigma)} \prod_{i=1}^{k} \delta_{l_i,l_i(\Sigma)}$$

Remark 2.3 [Residues]

The residue at $x = \alpha$ of a function f, picks the coefficient of the simple pole at $x = \alpha$, i.e. the coefficient of $(x - \alpha)^{-1}$ in the Laurent expansion of f in the vicinity of α. For instance the residue at $x = 0$ of a Laurent series $f(x)$ is

$$f(x) = \sum_k f_k x^k \quad \Rightarrow \quad \operatorname*{Res}_{x\to 0} f(x)dx = f_{-1}.$$

Through Cauchy's residue theorem [], the residue is a contour integral along a small circle $\mathcal{C}_\alpha$ encircling the pole α counterclockwise (and small circle means any circle small enough so that it doesn't encircle any other singularity of the integrand):

$$\operatorname*{Res}_{x\to\alpha} f(x)\, dx = \frac{1}{2i\pi} \oint_{\mathcal{C}_\alpha} f(x)\, dx\,,$$

therefore it makes sense to compute a residue of a differential form, for instance Res $f(x)dx$. In the literature, the dx is often omitted by abuse of notation, and one often writes Res $f(x)$, omitting the dx. This can be done only when there is no ambiguity on the integration variable, and the dx is implicitly assumed. It is particularly important to write the dx, when one wants to use changes of variables $x \to z$, and thus $dx = \frac{dx}{dz}\, dz$. Since changes of variables will play an important role in this book, we shall always write residues of differential forms.

Remark 2.4 When changing variable $x \to 1/x$, we have $d(1/x) = -dx/x^2$, and thus residues at ∞ come with a $-$ sign:

$$\operatorname*{Res}_{x\to\infty} \frac{1}{x}\, dx = -1$$

This is why we have the coefficients $(-1)^k$ in eq.(I-2-2).

Remark 2.5 differential forms:

One sees, for example from eq.(I-2-2), that $W_k^{(g)}(x_1, \ldots, x_k)$ will always be used to compute residues, i.e. integrals, and in fact, it will always appear together with $dx_1 \ldots dx_k$ as in $W_k^{(g)}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k$. The true nature of $W_k^{(g)}(x_1, \ldots, x_k)$, and of any generating series, is to be a differential form, and, anticipating on chapter VII, we claim that the fundamental intrinsic object is the differential form:

$$\omega_k^{(g)} = W_k^{(g)}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k.$$

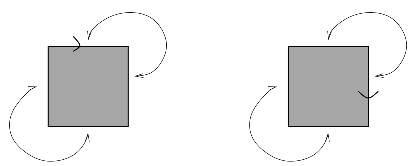
In this notation, $dx_1 \ldots dx_k$ is the tensor product of 1-forms $dx_1 \otimes \ldots \otimes dx_k$, it must not be confused with an exterior product $dx_1 \wedge \ldots \wedge dx_k$. It is symmetric in all dx_i's, not antisymmetric.

Examples:

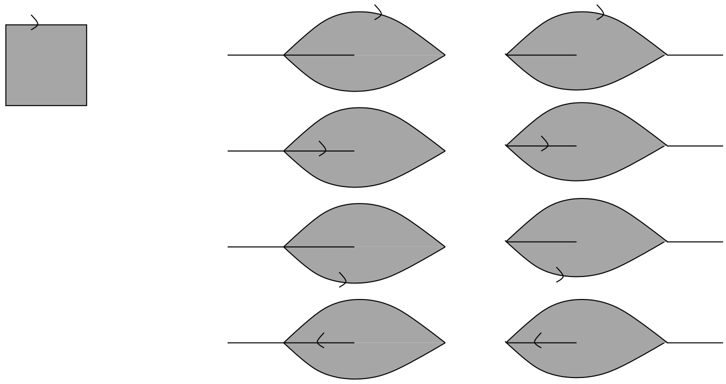
• If we choose all $t_j = 0$ except $t_4 \neq 0$, we count only quadrangulations, and $\mathcal{T}_4^{(g)}$ is the number of rooted quadrangulations of genus g, where all faces (including the one on the right of the marked edge) are quadrangles. The total number of faces is $n = n_4 + 1$, and the number of vertices is $v = n + 2 - 2g$. In the 60's, Tutte (this is the famous Tutte's formula [?, ?]) computed that (and we shall reserve it in chapter III):

$$\mathcal{T}_4^{(0)} = t^3 \sum_{n=1}^{\infty} (tt_4)^{n-1} \,\frac{2\,(2n)!\,3^n}{n!\,(n+2)!} = 2t^3 + 9t^4 t_4 + 54 t^5 t_4^2 + \ldots$$

The 2 maps of genus 0 with 1 marked quadrangle, 3 vertices and no unmarked quadrangles contributing to the term $2t^3$, are



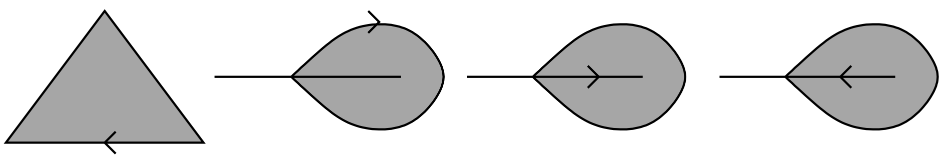
The 9 maps of genus 0 with 1 marked quadrangle, 4 vertices and 1 unmarked quadrangle, contributing to the term $9t^4 t_4$, are (where one face is the exterior face, and the marked face is the one on the right of the oriented marked edge)



• Similarly, if we choose all $t_j = 0$ except $t_3 \neq 0$, we count only triangulations, and $\mathcal{T}_3^{(g)}$ is the number of rooted triangulations of genus g, where all faces (including the one on the right of the marked edge) are triangles. The total number of faces is $2n = n_3 + 1$, and the number of vertices is $v = n + 2 - 2g$. In chapter III we compute explicitely:

$$\mathcal{T}_3^{(0)} = t^{5/2} \sum_{n=1}^{\infty} (t_3 \sqrt{t})^{2n-1}\, 2^{3n+1}\, \frac{\Gamma(\frac{3n}{2}+1)}{(n+2)!\,\Gamma(\frac{n}{2}+1)} = 4t^3 t_3 + 32 t^4 t_3^2 + \dots.$$

which was also computed by Tutte [?, ?]. The 4 maps of genus 0 with 1 marked triangle, 3 vertices and 1 unmarked triangle, contributing to the term $4t^3 t_3$, are (where one face is the exterior face, and the marked face is the one on the right of the oriented marked edge)



2.3 Redundancy of the parameters

One can remark that the number of vertices is redundant, because at fixed genus and boundaries, the number of vertices can be deduced from the numbers of polygonal faces. In other words the parameter t is redundant with the t_j's and x_j's.

Indeed, using eq.I-2-1: $v - (2-2g-k) = \frac{1}{2}\sum_{i=1}^k l_i + \frac{1}{2}\sum_{i=3}^d (i-2) n_i$, we may rewrite:

$$\begin{aligned} & t^{2g-2+k}\, W_k^{(g)}(x_1,\dots,x_k) dx_1 \dots dx_k \\ = \;& \delta_{k,1}\delta_{g,0}\frac{dx_1}{x_1} + \sum_{v=0}^{\infty} \sum_{\Sigma \in \mathbb{M}_k^{(g)}(v)} \frac{\prod_{i=3}^d (t_i\, t^{\frac{i}{2}-1})^{n_i(\Sigma)}}{\prod_{i=1}^k (x_i/\sqrt{t})^{l_i(\Sigma)}}\; \frac{1}{\#\mathrm{Aut}(\Sigma)}\; \prod_{i=1}^k \frac{dx_i}{x_i} \end{aligned}$$

This means that we can redefine:

$$t_i \to t_i\, t^{\frac{i}{2}-1} \qquad , \qquad x_i \to x_i/\sqrt{t} \qquad , \qquad t \to 1$$

and work with $t = 1$.

In other words, $W_k^{(g)}$ was defined as a formal power series in a single variable t (coupled to the number of vertices v), but we may also view it as a formal multiple power series in each t_i and x_i, coupled to the number of polygons n_i and degrees of boundaries l_i.

However, although t is a redundant parameter, we find more convenient to work with formal series in only one formal variable t than with multiple formal variables, and we shall keep t throughough this book.

2.4 All genus

It is convenient to define the generating function of maps regardless of their genus. Thanks to eq.(I-2-1), the number of vertices of a map is always such that $v+2-2g-k \geq 0$, and this allows to define the following formal power series of t:

$$W_k = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} W_k^{(g)} \tag{I-2-3}$$

one should be careful with this definition, as it is an equality between formal power series in powers of t, i.e. an equality between the coefficients of terms with equal powers of t. To any order in t, the sum over g is finite. Therefore the sum over g is **NOT a large N expansion**, it is a small t expansion.

Moreover, since any $\mathbb{M}_k^{(g)}(v)$ is a finite set, there is a maximal genus $g \leq g_{\max(v)}$ for each v, and the coefficients of W_k in powers of t are polynomials in $1/N$.

Similarly, for closed maps $k = 0$, we note $W_0^{(g)} = F_g$, and we can define the all genus generating function of closed maps:

$$F = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} F_g. \tag{I-2-4}$$

Remark 2.6 We will see later in this book, that each series $W_k^{(g)}$ has a finite radius of convergency, and is in fact an algebraic function. But the all genus generating function W_k is not algebraic, and may have a vanishing radius of convergency.

2.5 Non connected maps

When we have a formal generating series counting disconnected objects multiplicatively, it is well known that the log is the generating function which counts only the connected objects, see []. If we denote $Z_N(t_3,\dots,t_d;t)$ the generating function for closed maps not necessarily connected, we have:

$$\ln\left(Z_N(t_3,t_4,\dots,t_d;t)\right) = F(t_3,\dots,t_d;t;N) \qquad \text{(I-2-5)}$$

where again this equality is to be taken as an equality between formal series, i.e. equality between the coefficients in the small t expansion.

For open maps with $k\geq 1$ boundaries, there are several ways of obtaining disconnected surfaces, because each disconnected piece may carry either no boundary, or subsets of the set of boundaries. The generating functions of connected objects are **cumulants** of the non-connected ones.

Let $W_k^*(x_1,\dots,x_k)$ be the generating function of not-necessarily connected maps of all genus. We have:

$$W_1^*(x_1) = Z_N\ W_1(x_1)$$

$$W_2^*(x_1,x_2) = Z_N\ \left(W_2(x_1,x_2) + W_1(x_1)\, W_1(x_2)\right)$$

$$\begin{aligned} W_3^*(x_1,x_2,x_3) &= Z_N\ (W_3(x_1,x_2) + W_1(x_1)\, W_2(x_2,x_3) + W_1(x_2)\, W_2(x_1,x_3) \\ &\quad + W_1(x_3)\, W_2(x_1,x_2) + W_1(x_1)\, W_1(x_2)\, W_1(x_3)) \end{aligned}$$

and so on, if we note $K=\{x_1,\dots,x_k\}$:

$$W_k^*(K) = Z_N \sum_{n=1}^k \sum_{J_1\uplus\dots\uplus J_n=K} \prod_{i=1}^n W_{\#J_i}(J_i)$$

where we sum over all possible partitions of K.

The converse is called **cumulants**, or sometimes **connected parts**:

$$W_1(x_1) = \frac{1}{Z_N}\ W_1^*(x_1)$$

$$W_2(x_1,x_2) = \frac{W_2^*(x_1,x_2)}{Z_N} - \frac{W_1^*(x_1)}{Z_N}\,\frac{W_1^*(x_2)}{Z_N}$$

$$\begin{aligned} W_3(x_1,x_2,x_3) &= \frac{W_3^*(x_1,x_2,x_3)}{Z_N} - \frac{W_1^*(x_1)}{Z_N}\,\frac{W_2^*(x_2,x_3)}{Z_N} - \frac{W_1^*(x_2)}{Z_N}\,\frac{W_2^*(x_1,x_3)}{Z_N} \\ &\quad - \frac{W_1^*(x_3)}{Z_N}\,\frac{W_2^*(x_1,x_2)}{Z_N} + 2\,\frac{W_1^*(x_1)}{Z_N}\,\frac{W_1^*(x_2)}{Z_N}\,\frac{W_1^*(x_3)}{Z_N} \end{aligned}$$

and so on...

2.6 Rooted maps: one boundary

The case $k=1$, i.e. one boundary plays a special role.

A map with one boundary, is also, by definition, a map with one oriented marked edge, it is also called a **rooted map**. The marked face is the face on the right of the oriented marked edge.

A reason of the special role of maps with one boundary, is that there is only one automorphism (the identity) which can conserve the map and the oriented marked edge. Therefore if $k = 1$ we have

$$k = 1 \quad \Rightarrow \quad \#\mathrm{Aut} = 1$$

A planar map with one boundary is called a **"disk"**.

3 Tutte's equations

3.1 Planar case: the disk

The Canadian mathematician Tutte discovered a combinatoric recursive equation in 1963 [**?**], for counting planar maps with one boundary, i.e. rooted planar maps, or equivalently disks.

The idea is the following. A planar map with one boundary of length $l + 1$, is in fact a planar map with one marked face of degree $l+1$, with one marked oriented edge on it. Let us draw the map on the projective plane, such that ∞ is in the marked face, i.e. the marked face is on the exterior. The marked edge, separates two faces (not necessarily distinct).

If one removes the marked edge, two situations may occur:

• either the face on the other side of the marked edge was the same, thus if we remove the edge, we get two planar maps, each having one boundary (we mark the edges adjacent to the removed edge), one has a boundary of length j, the other $l-1-j$, for some j such that $0 \leq j \leq l-1$.

• either the face on the other side is not the same, and thus it is an unmarked face of degree j for some j such that $3 \leq j \leq d$. When we remove the marked edge, the map remains connected, and we get a new map, with a boundary of length $l + j - 1$ (the new marked edge is the one adjacent to the removed edge). Removing the edge was thus equivalent to removing a j-gone, which has weight t_j.

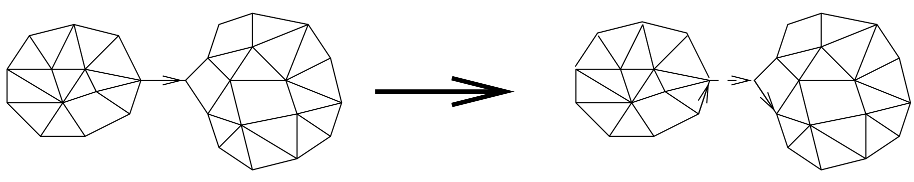
It is easy to see that this procedure of removing the marked edge is a bijection

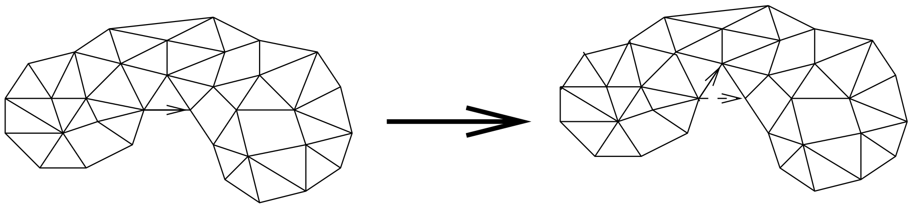
$$\mathbb{M}^{(0)}_{1;l+1} \to \cup_{j=0}^{l-1} \mathbb{M}^{(0)}_{1;j} \times \mathbb{M}^{(0)}_{1;l-1-j} + \cup_{j=3}^{d} \mathbb{M}^{(0)}_{1;l+j-1}$$

and thus Tutte's equation follows:

$$\boxed{\mathcal{T}^{(0)}_{l+1} = \sum_{j=0}^{l-1} \mathcal{T}^{(0)}_{j} \mathcal{T}^{(0)}_{l-1-j} + \sum_{j=3}^{d} t_j \mathcal{T}^{(0)}_{l+j-1}} \tag{I-3-1}$$

Tutte's proof is illustrated as follows:





3.2 Higher genus case

A similar recursive equation can be found for higher genus or higher number of boundaries.

Consider a discrete surface of genus g, with k boundaries of respective lengths $l_1+1, l_2, \dots, l_k$, and let us denote collectivelly

$$K = \{l_2, \dots, l_k\}$$

Then we erase the marked edge of the first boundary. Several mutually exclusive possibilities can occur:

- the marked edge separates the marked face with some unmarked face (let us say a j-gone with $3 \leq j \leq d$), and removing that edge is equivalent to removing a j-gone (with weight t_j), and we thus get a map of genus g with the same number of boundaries, and the length of the first boundary is now $l_1 + j - 1$.

- the marked edge separates two distinct marked faces (face 1 and face m with $2 \leq m \leq k$,), thus the marked edge of the first boundary is one of the l_m edges of the m^{th} boundary. We thus get a map of genus g with $k-1$ boundaries. the other $k-2$ boundaries remain unchanged, and there is now one boundary of length $l_1 + l_m - 1$.

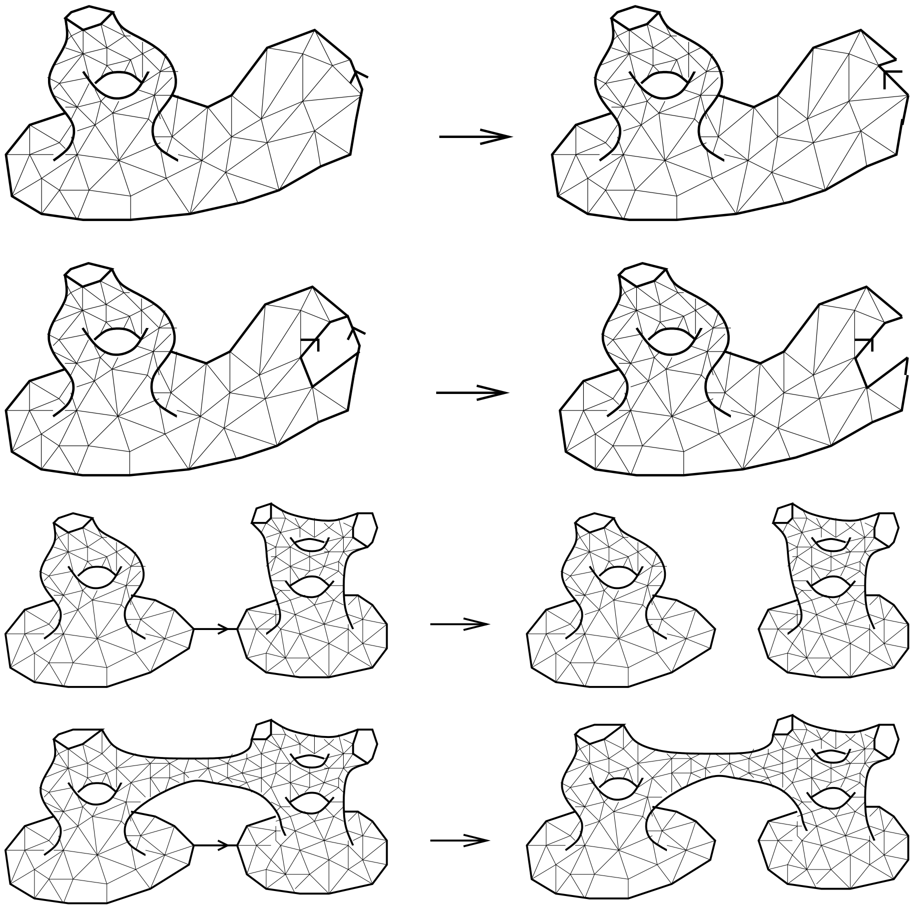
- the same marked face lies on both sides of the marked edge, therefore by removing it, we disconnect the boundary. Two cases can occur: either the map itself gets disconnected into two maps of genus h and $g-h$, one having $1 + \#J$ boundaries of lengths (j, J), where J is a subset of K, and the other map having $k - \#J$ boundaries of lengths $(l_1 - 1 - j, K \setminus J)$, or the map remains connected because there was a handle connecting the two sides, and thus by removing the marked edge, we get a map of genus $g-1$, with $k+1$ boundaries of lengths $(j, l_1 - j - 1, K)$.

Again, this procedure is (up to the symmetry factors) bijective, and all those possibilities correspond to the following recursive equation:

$$\begin{aligned}
&\sum_{j=0}^{l_1-1} \Big[\sum_{h=0}^{g} \sum_{J \subset K} \mathcal{T}^{(h)}_{j,J} \mathcal{T}^{(g-h)}_{l_1-1-j, K\setminus J} + \mathcal{T}^{(g-1)}_{j, l_1-1-j, K} \Big] + \sum_{m=2}^{k} l_m\, \mathcal{T}^{(g)}_{l_m+l_1-1, K\setminus\{l_m\}} \\
&= \mathcal{T}^{(g)}_{l_1+1,K} - \sum_{j=3}^{d} t_j \mathcal{T}^{(g)}_{l_1+j-1,K} \qquad \text{(I-3-2)}
\end{aligned}$$

which we call "Loop equation" or "higher genus Tutte's equation".

This equation is illustrated as follows:



Here, we have presented only an intuitive derivation, and we present a more rigorous derivation in chapter III, with a very different technique, called loop equations for formal matrix integrals.

4 Bibliography

Several classical books exist about maps, for instance: C. Berge [?], W.T. Tutte [?, ?], J.L. Gross, T.W. Tucker [?], as well as [?, ?, ?, ?].

5 Exercises

Exercise 1:

Count all connected quadrangulations with $n_4 = 1$ and $n_4 = 2$ quadrangles, count them with their symmetry factors and according to their topology.

answer: There is 1 planar quadrangulation with $n_4 = 1$ quadrangle, and it has symmetry factor 2, and 1 quadrangulation of genus $g = 1$ with $n_4 = 1$ quadrangle, and it has symmetry factor 4.

There are 2 planar quadrangulations with $n_4 = 2$ quadrangles, one has symmetry factor 8, one has symmetry factor 1. And there are 4 quadrangulations of genus $g = 1$ with $n_4 = 2$ quadrangles, one has symmetry factor 8, one has symmetry factor 4, one has symmetry factor 2, one has symmetry factor 1.

This can be summarized as:

$$F = t\, t_4 \left(\frac{N^2}{2} + \frac{1}{4}\right) + t^2\, t_4^2 \left(N^2 \left(1 + \frac{1}{8}\right) + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1\right) + O(t_4^3).$$

Exercise 2:

Find all planar maps with 1 marked face of arbitrary length l, and whose unmarked faces are only pentagons, and with up to 5 vertices.

Check that

$$W_1^{(0)} = \frac{t}{x} + \frac{t^2}{x^3} + t^3 \left(\frac{2}{x^5} + \frac{2t_5}{x^2}\right) + t^4 \left(\frac{5}{x^7} + \frac{9t_5}{x^4}\right) + t^5 \left(\frac{12}{x^9} + \frac{17t_5}{x^6} + \frac{3t_5^2}{x^3}\right) + O(t^6)$$